Chapter 2
Section 1
Chapter Objectives

- Solve linear equations.
- Solve a formula for a specified variable.
- Solve application problems using linear equations.
2.1 Linear Equations in One Variable

1. Decide whether a number is a solution of a linear equation.
2. Solve linear equations by using the addition and multiplication properties of equality.
3. Solve linear equations by using the distributive property.
4. Solve linear equations with fractions or decimals.
5. Identify conditional equations, contradictions, and identities.
Equations and inequalities compare algebraic expressions.

*An equation always contains an equals sign, while an expression does not.*

\[ 3x - 7 = 2 \]

**Left side** \[ 3x - 7 \]

**Right side** \[ 3x - 7 \]

Equation (to solve)  
Expression (to simplify or evaluate)
A linear equation in one variable can be written in the form

\[ Ax + B = C, \]

where \( A, B, \) and \( C \) are real numbers, with \( A \neq 0. \)

A linear equation is a first-degree equation, since the greatest power on the variable is 1.
EXAMPLE

Decide whether each of the following is an equation or expression.

a. $9x = 10$  
   equation

b. $9x + 10$  
   expression

c. $3 + 5x - 8x + 9$  
   expression

d. $3 + 5x = -8x + 9$  
   equation
Objective 1

Decide whether a number is a solution of a linear equation.
If the variable in an equation can be replaced by a real number that makes the statement true, then that number is a solution of the equation.

An equation is solved by finding its solution set, the set of all solutions.

Equivalent equations are related equations that have the same solution set.
Objective

Solve linear equations by using the addition and multiplication properties of equality.
Addition and Multiplication Properties of Equality

**Addition Property of Equality**
For all real numbers $A$, $B$, and $C$, the equations

\[ A = B \quad \text{and} \quad A + C = B + C \]

are equivalent.

That is, the same number may be added to each side of an equation without changing the solution set.

**Multiplication Property of Equality**
For all real numbers $A$, and $B$, and for $C \neq 0$, the equations

\[ A = B \quad \text{and} \quad AC = BC \]

are equivalent.

That is, each side of the equation may be multiplied by the same nonzero number without changing the solution set.
EXAMPLE 1

Solve $4x + 8x = -9 + 17x - 1$.

The goal is to isolate $x$ on one side of the equation.

\[ 4x + 8x = -9 + 17x - 1 \]
\[ 12x = -10 + 17x \quad \text{Combine like terms.} \]
\[ 12x - 17x = -10 + 17x - 17x \quad \text{Subtract 17x from each side.} \]
\[ -5x = -10 \quad \text{Combine like terms.} \]
\[ \frac{-5x}{-5} = \frac{-10}{-5} \quad \text{Divide each side by } -5. \]
\[ x = 2 \]

Check $x = 2$ in the original equation.
continued

Check \( x = 2 \) in the original equation.

\[ 4x + 8x = -9 + 17x - 1 \]

\[ 4(2) + 8(2) = -9 + 17(2) - 1 \]

\[ 8 + 16 = -9 + 34 - 1 \]

\[ 24 = 24 \]

The true statement indicates that \( \{2\} \) is the solution set.

Use parentheses around substituted values to avoid errors.

This is NOT the solution.
Solving a Linear Equation in One Variable

**Step 1**  Clear fractions. Eliminate any fractions by multiplying each side by the least common denominator.

**Step 2**  Simplify each side separately. Use the distributive property to clear parentheses and combine like terms as needed.

**Step 3**  Isolate the variable terms on one side. Use the addition property to get all terms with variables on one side of the equation and all numbers on the other.

**Step 4**  Isolate the variable. Use the multiplication property to get an equation with just the variable (with coefficient 1) on one side.

**Step 5**  Check. Substitute the proposed solution into the original equation.
Objective 3

Solve linear equations by using the distributive property.
EXAMPLE 2

Solve \(6 - (4 + m) = 8m - 2(3m + 5)\).

**Step 1** Since there are no fractions in the equation, Step 1 does not apply.

**Step 2** Use the distributive property to simplify and combine like terms on the left and right.

\[
6 - (4 + m) = 8m - 2(3m + 5)
\]

\[
6 - 1 \cdot 4 - 1 \cdot m = 8m - 2 \cdot 3m + (-2) \cdot 5
\]

\[
6 - 4 - m = 8m - 6m - 10
\]
6 – 4 – m = 8m – 6m – 10

2 – m = 2m – 10

**Step 3**  Next, use the addition property of equality.

2 – 2 – m = 2m – 10 – 2

–m = 2m – 12

–m – 2m = 2m – 2m – 12

–3m = –12

**Step 4**  Use the multiplication property of equality to isolate m on the left side. –3m = –12

\[
\begin{align*}
-3 & \div -3 \\
m & = 4
\end{align*}
\]
Step 5 Check: $6 - (4 + m) = 8m - 2(3m + 5)$

$6 - (4 + 4) = 8(4) - 2(3(4) + 5)$

$6 - 8 = 32 - 2(12 + 5)$

$-2 = 32 - 2(17)$

$-2 = 32 - 34$

$-2 = -2$ True

The solution checks, so $\{4\}$ is the solution set.
Objective 4

Solve linear equations with fractions and decimals.
EXAMPLE 3

Solve \( \frac{k + 1}{2} + \frac{k + 3}{4} = \frac{1}{2} \).

Start by eliminating the fractions. Multiply both sides by the LCD, 4.

**Step 1**

\[
4 \left( \frac{k + 1}{2} + \frac{k + 3}{4} \right) = 4 \left( \frac{1}{2} \right)
\]

**Step 2**

\[
4 \left( \frac{k + 1}{2} \right) + 4 \left( \frac{k + 3}{4} \right) = 4 \left( \frac{1}{2} \right) \quad \text{Distributive property.}
\]

\[
\frac{4(k + 1)}{2} + \frac{4(k + 3)}{4} = 2 \quad \text{Multiply; 4.}
\]
continued

\[
\frac{4(k + 1)}{2} + \frac{4(k + 3)}{4} = 2
\]

\[2(k + 1) + k + 3 = 2\]

\[2(k) + 2(1) + k + 3 = 2\]  \hspace{1cm} \text{Distributive property.}

\[2k + 2 + k + 3 = 2\]  \hspace{1cm} \text{Multiply; 4.}

\[3k + 5 = 2\]  \hspace{1cm} \text{Combine like terms.}

\textbf{Step 3}\hspace{1cm}3k + 5 - 5 = 2 - 5\hspace{1cm} \text{Subtract 5.}

\[3k = -3\]  \hspace{1cm} \text{Combine like terms.}

\textbf{Step 4}\hspace{1cm}\frac{3k}{3} = \frac{-3}{3}\hspace{1cm} \text{Divide by 3.}

\[k = -1\]
Step 5

Check: \[
\frac{(k + 1)}{2} + \frac{(k + 3)}{4} = \frac{1}{2}
\]

\[
\frac{(k + 1)}{2} + \frac{(k + 3)}{4} = \frac{1}{2}
\]

\[
\frac{(-1 + 1)}{2} + \frac{(-1 + 3)}{4} = \frac{1}{2}
\]

\[
\frac{0}{2} + \frac{2}{4} = \frac{1}{2}
\]

\[
\frac{1}{2} = \frac{1}{2}
\]

The solution checks, so the solution set is \{-1\}. 
EXAMPLE 4

Solve \(0.02(60) + 0.04p = 0.03(50 + p)\).

\[
0.02(60) + 0.04p = 0.03(50 + p)
\]

\[
2(60) + 4p = 3(50 + p)
\]

\[
120 + 4p = 150 + 3p
\]

\[
120 - 120 + 4p = 150 - 120 + 3p
\]

\[
4p = 30 + 3p
\]

\[
4p - 3p = 30 + 3p - 3p
\]

\[
p = 30
\]

Since each decimal number is given in hundredths, multiply both sides of the equation by 100.
Objective 5

Identify conditional equations, contradictions, and identities.
<table>
<thead>
<tr>
<th><strong>Type of Linear Equation</strong></th>
<th><strong>Number of Solutions</strong></th>
<th><strong>Indication when Solving</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Conditional</strong></td>
<td>One</td>
<td>Final line is $x = a$ number.</td>
</tr>
<tr>
<td><strong>Identity</strong></td>
<td>Infinite; solution set {all real numbers}</td>
<td>Final line is true, such as $0 = 0$.</td>
</tr>
<tr>
<td><strong>Contradiction</strong></td>
<td>None; solution set $\emptyset$</td>
<td>Final line is false, such as $-15 = -20$.</td>
</tr>
</tbody>
</table>
EXAMPLE 5

Solve each equation. Decide whether it is a conditional equation, an identity, or a contradiction.

a. \(5(x + 2) - 2(x + 1) = 3x + 1\)

\[
5x + 10 - 2x - 2 = 3x + 1
\]

\[
3x + 8 = 3x + 1
\]

\[
3x - 3x + 8 = 3x - 3x + 1
\]

\[
8 = 1 \quad \text{False}
\]

The result is false, the equation has no solution. The equation is a contradiction. The solution set is \(\emptyset\).
Solve each equation. Decide whether it is a **conditional equation**, an **identity**, or a **contradiction**.

b. \[ \frac{x + 1}{3} + \frac{2x}{3} = x + \frac{1}{3} \]

Multiply each side by the LCD, 3.

\[
3\left(\frac{x + 1}{3}\right) + 3\left(\frac{2x}{3}\right) = 3\left(x + \frac{1}{3}\right)
\]

\[x + 1 + 2x = 3x + 1\]

\[3x + 1 = 3x + 1\]

This is an identity. Any real number will make the equation true. The solution set is \{all real numbers\}. 
Solve each equation. Decide whether it is a conditional equation, an identity, or a contradiction.

c. $5(3x + 1) = x + 5$

$15x + 5 = x + 5$

$15x - x + 5 = x - x + 5$

$14x + 5 = 5$

$14x + 5 - 5 = 5 - 5$

$14x = 0$

$x = 0$

This is a conditional equation. The solution set is $\{0\}$. 

continued
Formulas

1. Solve a formula for a specified variable.
2. Solve applied problems by using formulas.
3. Solve percent problems.
Objective 1

Solve a formula for a specified variable.
A **mathematical model** is an equation or inequality that describes a real situation. Models for many applied problems already exist; they are called *formulas*. A *formula* is the an equation in which variables are used to describe a relationship.

Some formulas are

\[ d = rt, \quad I = prt, \quad \text{and} \quad P = 2L + 2W. \]
Solving for a Specified Variable

**Step 1** Transform so that all terms containing the specified variable are on one side of the equation and all terms without that variable are on the other side.

**Step 2** If necessary, use the distributive property to combine the terms with the specified variable.* The result should be the product of the sum or difference and the variable.

**Step 3** Divide both sides by the factor that is the coefficient of the specified variable.
EXAMPLE 1

Solve $m = 2k + 3b$ for $k$.

Solve the formula by isolating the $k$ on one side of the equals sign.

$$m = 2k + 3b$$

**Step 1**

$$m - 3b = 2k + 3b - 3b$$

Subtract 3b.

$$m - 3b = 2k$$

**Step 2**

$$\frac{m - 3b}{2} = \frac{2k}{2}$$

Divide by 2.

**Step 3**

$$\frac{m - 3b}{2} = k \quad \text{or} \quad k = \frac{m - 3b}{2}$$
EXAMPLE 2

Solve the formula \( y = \frac{1}{2}(x + 3) \) for \( x \).

\[
y = \frac{1}{2}(x + 3)
\]

\[
2y = x + 3 \quad \text{Multiply by 2.}
\]

\[
2y - 3 = x \quad \text{or} \quad x = 2y - 3 \quad \text{Subtract 3.}
\]
EXAMPLE 3

Solve the formula \( A = 2HW + 2LW + 2LH \) for \( W \).

\[
A = 2HW + 2LW + 2LH
\]

\[
A - 2LH = 2HW + 2LW \quad \text{Subtract } 2LH.
\]

\[
A - 2LH = W(2H + 2L) \quad \text{Distributive property}
\]

\[
\frac{A - 2LH}{2H + 2L} = \frac{W(2H + 2L)}{2H + 2L}
\]

\[
\frac{A - 2LH}{2H + 2L} = W, \quad \text{or} \quad W = \frac{A - 2LH}{2H + 2L}
\]
CAUTION  The most common error in working a problem like in Example 3 is not using the distributive property correctly. We must write the expression so that the specified variable is a *factor*; then we can divide by its coefficient in the final step.
Objective 2

Solve applied problems by using formulas.
EXAMPLE 4

The distance is 500 mi and the rate is 25 mph. Find the time.

Find the formula for time by solving \( d = rt \) for \( t \).

\[
d = rt
\]

\[
\frac{d}{r} = \frac{rt}{r}
\]

Divide by \( r \).

\[
\frac{d}{r} = t \quad \text{or} \quad t = \frac{d}{r}
\]
continued

Now substitute $d = 500$ and $r = 25$.

\[
t = \frac{d}{r}
\]

\[
t = \frac{500}{25}
\]

Let $d = 500$, $r = 25$.

Divide.

\[
t = 20
\]

The time is 20 hours.
PROBLEM-SOLVING HINT  As seen in Example 4, it may be convenient to first solve for a specified unknown variable before substituting the given values. This is particularly useful when we wish to substitute several different values for the same variable. For example, an economics class might need to solve the equation $I = prt$ for $r$ to find rates that produce specified amounts of interest for various principals and times.
Objective

Solve percent problems.
An important everyday use of mathematics involves the concept of **percent**. Percent is written with the symbol %. The word percent means “per one hundred”.

\[ 1\% = 0.01 \quad \text{or} \quad 1\% = \frac{1}{100} \]

The following formula can be used to solve a percent problem:

\[
\frac{\text{partial amount}}{\text{whole amount}} = \text{percent} \quad \text{(represented as a decimal)}.
\]
EXAMPLE 5

Solve each problem.

a. A mixture of gasoline oil contains 20 oz, of which 1 oz is oil. What percent of the mixture is oil?

The given amount of mixture is 20 oz. The part that is oil is 1 oz. thus, the percent of oil is

\[
x = \frac{1}{20} \quad \text{partial amount}
\]

\[
x = 0.05, \quad \text{or} \quad 5%.
\]

Thus, 5% of the mixture is oil.
b. An automobile salesman earns an 8% commission on every car he sells. How much does he earn on a car that sells for $12,000?

Let $x$ represent the amount of commission earned.

\[ 8\% = 8 \cdot 0.01 = 0.08 \]

\[ \frac{x}{12,000} = 0.08 \]

\[ x = 0.08(12,000) \]

\[ x = 960 \]

The salesman earns $960.
EXAMPLE 6

In 2005, people in the United States spent an estimated $35.9 billion on their pets. How much was spent on pet supplies/medicine? Round your answer to the nearest tenth of a billion dollars.

Let \( x \) represent the amount spent on pet supplies/medicine.

\[
\frac{x}{35.9} = 0.245
\]

\[
x = 0.245(35.9)
\]

\[
x = 8.7955
\]

Therefore, about $8.8 billion was spent on pet supplies/medicine.
Applications of Linear Equations

1. Translate from words to mathematical expressions.
2. Write equations from given information.
3. Distinguish between expressions and equations.
4. Use the six steps in solving an applied problem.
5. Solve percent problems.
7. Solve mixture problems.
Objective 1

Translate from words to mathematical expressions.
<table>
<thead>
<tr>
<th>Verbal Expression</th>
<th>Mathematical Expression (where x and y are numbers)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Addition</strong></td>
<td></td>
</tr>
<tr>
<td>The <strong>sum</strong> of a number and 7</td>
<td>( x + 7 )</td>
</tr>
<tr>
<td>6 more than a number</td>
<td>( x + 6 )</td>
</tr>
<tr>
<td>3 plus a number</td>
<td>( 3 + x )</td>
</tr>
<tr>
<td>24 added to a number</td>
<td>( x + 24 )</td>
</tr>
<tr>
<td>A number <strong>increased by</strong> 5</td>
<td>( x + 5 )</td>
</tr>
<tr>
<td>The <strong>sum</strong> of two numbers</td>
<td>( x + y )</td>
</tr>
<tr>
<td>Verbal Expression</td>
<td>Mathematical Expression (where (x) and (y) are numbers)</td>
</tr>
<tr>
<td>----------------------------------------------</td>
<td>-------------------------------------------------------------</td>
</tr>
<tr>
<td><strong>Subtraction</strong></td>
<td></td>
</tr>
<tr>
<td>2 less than a number</td>
<td>(x - 2)</td>
</tr>
<tr>
<td>2 less a number</td>
<td>(2 - x)</td>
</tr>
<tr>
<td>12 minus a number</td>
<td>(12 - x)</td>
</tr>
<tr>
<td>A number decreased by 12</td>
<td>(x - 12)</td>
</tr>
<tr>
<td>A number subtracted from 10</td>
<td>(10 - x)</td>
</tr>
<tr>
<td>From a number, subtract 10</td>
<td>(x - 10)</td>
</tr>
<tr>
<td>The difference between two numbers</td>
<td>(x - y)</td>
</tr>
<tr>
<td>Verbal Expression</td>
<td>Mathematical Expression (where x and y are numbers)</td>
</tr>
<tr>
<td>---------------------------------------</td>
<td>----------------------------------------------------</td>
</tr>
<tr>
<td><strong>Multiplication</strong></td>
<td></td>
</tr>
<tr>
<td>16 times a number</td>
<td>$16x$</td>
</tr>
<tr>
<td>A number multiplied by 6</td>
<td>$6x$</td>
</tr>
<tr>
<td>2/3 of a number (used with fractions and percent)</td>
<td>$\frac{2}{3}x$</td>
</tr>
<tr>
<td>3/4 as much as a number</td>
<td>$\frac{3}{4}x$</td>
</tr>
<tr>
<td>Twice (2 times) a number</td>
<td>$2x$</td>
</tr>
<tr>
<td>The product of two numbers</td>
<td>$xy$</td>
</tr>
<tr>
<td><strong>Verbal Expression</strong></td>
<td><strong>Mathematical Expression (where x and y are numbers)</strong></td>
</tr>
<tr>
<td>-----------------------</td>
<td>------------------------------------------------------</td>
</tr>
<tr>
<td><strong>Division</strong></td>
<td></td>
</tr>
<tr>
<td>The quotient of 8 and a number</td>
<td>$\frac{8}{x} \ (x \neq 0)$</td>
</tr>
<tr>
<td>A number divided by 13</td>
<td>$\frac{x}{13}$</td>
</tr>
<tr>
<td>The ratio of two numbers or the quotient of two numbers</td>
<td>$\frac{x}{y} \ (y \neq 0)$</td>
</tr>
</tbody>
</table>
Objective

Write equations from given information.
EXAMPLE 1

Translate each verbal sentence into an equation, using $x$ as the variable.

a. The sum of a number and 6 is 28.  \[ x + 6 = 28 \]

b. The product of a number and 7 is twice the number plus 12.  \[ 7x = 2x + 12 \]

c. The quotient of a number and 6, added to twice the number is 7.  \[ 2x + \frac{x}{6} = 7 \]
Objective 3

Distinguish between expressions and equations.
EXAMPLE 2

Decide whether each is an expression or equation.

a. $5x - 3(x + 2) = 7$

There is an equals sign with something on either side of it, this is an equation.

b. $5x - 3(x + 2)$

There is no equals sign, so this is an expression.
Objective 4

Use the six steps in solving an applied problem.
Solving an Applied Problem

**Step 1** Read the problem, several times if necessary, until you understand what is given and what is to be found.

**Step 2** Assign a variable to represent the unknown value, using diagrams or tables as needed. Write down what the variable represents. If necessary, express any other unknown values in terms of the variable.

**Step 3** Write an equation using the variable expression(s).

**Step 4** Solve the equation.

**Step 5** State the answer to the problem. Does it seem reasonable?

**Step 6** Check the answer in the words of the original problem.
EXAMPLE 3

The length of a rectangle is 5 cm more than its width. The perimeter is five times the width. What are the dimensions of the rectangle?

**Step 1** Read the problem. What must be found?

*The length and width of the rectangle.*

What is given?

*The length is 5 cm more than its width; the perimeter is 5 times the width.*

**Step 2** Assign a variable. Let \( w = \) the width; then \( w + 5 = \) the length. Make a sketch.
Continued

**Step 3  Write an equation.** Use the formula for the perimeter of a rectangle.

\[ P = 2l + 2w \]

\[ 5w = 2(w + 5) + 2(w) \]

**Step 4  Solve the equation.**

\[ 5w = 2w + 10 + 2w \]

\[ 5w = 4w + 10 \]

\[ 5w - 4w = 4w - 4w + 10 \]

\[ w = 10 \]
Step 5  State the answer. The width of the rectangle is 10 cm and the length is $10 + 5 = 15$ cm.

Step 6  Check.

The perimeter is 5 times the width.

$P = 2l + 2w$

$5w = 2(15) + 2(10)$

$50 = 30 + 20$

$50 = 50$

The solution checks.
EXAMPLE 4

For the 2005 baseball season, the Major League Baseball leaders in runs batted in (RBIs) were Andruw Jones of the Atlanta Braves in the National League and David Ortiz of the Boston Red Sox in the American League. These two players had a total of 276 RBIs, and Ortiz had 20 more than Jones. How many RBIs did each player have?

Step 1 Read the problem. We are asked to find the number of RBIs each player had.
continued

Step 2 Assign a variable.
Let $x$ represent the number of RBIs for Jones.
Let $x + 20$ represent the RBIs for Ortiz

Step 3 Write an equation.
The sum of the RBI’s is 276.
$x + x + 20 = 276$

Step 4 Solve the equation.
$x + x + 20 = 276$

\[ 2x + 20 = 276 \]

\[ 2x + 20 - 20 = 276 - 20 \]
\[ 2x = 256 \]
\[ x = 128 \]
**continued**

**Step 4** \(x = 128\)

**Step 5  State the answer.** We let \(x\) represent the number of RBIs for Jones.

Then Ortiz has \(x + 20 = 128 + 20 = 148\)

**Step 6  Check.**

148 is 20 more than 128, and \(148 + 128 = 276\). The conditions of the problem are satisfied, and our answer checks.
Objective 5

Solve percent problems.
EXAMPLE 5

187.5 is 125% of some number. What is that number?

Step 1  Read the problem. We are given 187.5 is 125% of some number. We are asked to find the number.

Step 2  Assign a variable.

Let $x = \text{the number}$

125% = 1.25

Step 3  Write an equation from the given information.

187.5 is 125% of some number

$187.5 = 1.25x$
**Step 4** Solve the equation.

\[ 187.5 = 1.25x \]

\[ \frac{187.5}{1.25} = \frac{1.25x}{1.25} \]

\[ 150 = x \]

**Step 5** State the answer. 187.5 is 125% of 150.

**Step 6** Check.

\[ 150 \cdot 125\% = 187.5 \] (the answer checks)
Objective 6

Solve investment problems.
EXAMPLE 6

A man has $34,000 to invest. He invests some of the money at 5% and the balance at 4%. His total annual interest income is $1545. Find the amount invested at each rate.

Step 1  Read the problem. We must find the two amounts.

Step 2  Assign a variable.

Let \( x \) = the amount to invest at 5%

\( 34,000 - x \) = the amount to invest at 4%
Step 2  Assign a variable.

The formula for interest is $I = prt$. Here the time is 1 yr. Use a table to organize the given information.

<table>
<thead>
<tr>
<th>Principal</th>
<th>Rate (as a decimal)</th>
<th>Interest</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$</td>
<td>0.05</td>
<td>$0.05x$</td>
</tr>
<tr>
<td>$34,000 - x$</td>
<td>0.04</td>
<td>$0.04(34,000 - x)$</td>
</tr>
<tr>
<td>$34,000$</td>
<td>XXXXXXXXXX</td>
<td>1545</td>
</tr>
</tbody>
</table>

Step 3  Write an equation. The last column of the table gives the equation.

\[
0.05x + 0.04(34,000 - x) = 1545
\]
Step 4 Solve the equation. We do so without clearing decimals.

\[0.05x + 0.04(34,000 - x) = 1545\]

\[0.05x + 1360 - 0.04x = 1545\]

\[0.01x + 1360 = 1545\]

\[0.01x = 185\]

\[x = 18,500\]

Step 5 State the answer. $18,500 was invested at 5% and 15,500 was invested at 4%.

Step 6 Check by finding the annual interest at each rate.

\[0.05(18,500) = 925\]

\[0.04(15,500) = 620\]

\[925 + 620 = 1545\]
Objective

Solve mixture problems.
EXAMPLE 7

How many pounds of candy worth $8 per lb should be mixed with 100 lb of candy worth $4 per lb to get a mixture that can be sold for $7 per lb?

Step 1  Read the problem. The problem asks for the amount of candy worth $8 to be used.

Step 2  Assign a variable. Let $x = $ the amount of $8 candy

<table>
<thead>
<tr>
<th>Number of pounds</th>
<th>$ Amount</th>
<th>Pounds of Candy worth $7</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>$4</td>
<td>100(4) = 400</td>
</tr>
<tr>
<td>$x</td>
<td>$8</td>
<td>$8x</td>
</tr>
<tr>
<td>100 + $x</td>
<td>$7</td>
<td>7(100 + $x)</td>
</tr>
</tbody>
</table>
continued

Step 3  Write an equation.

\[ 400 + 8x = 7(100 + x) \]

Step 4  Solve.

\[ 400 + 8x = 700 + 7x \]

\[ x = 300 \]

Step 5  State the answer. 300 pounds of candy worth $8 per pound should be used.

Step 6  Check.

300 lb worth $8 + 100 lb worth $4 = $7(100 + 300)

\[ $2400 + $400 = $7(400) \]

\[ $2800 = $2800 \]
EXAMPLE 8

How much water must be added to 20 L of 50% antifreeze solution to reduce it to 40% antifreeze?

Step 1  Read the problem. The problem asks for the amount of pure water to be added.

Step 2  Assign a variable. Let $x = \text{the number of liters of pure water}$

<table>
<thead>
<tr>
<th>Number of liters</th>
<th>Percent (as a decimal)</th>
<th>Liters of Pure Water</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>20</td>
<td>0.5</td>
<td>20(0.5)</td>
</tr>
<tr>
<td>$x + 20$</td>
<td>0.4</td>
<td>0.4($x + 20$)</td>
</tr>
</tbody>
</table>
Step 3 Write an equation.

\[ 0 + 20(0.5) = 0.4(x + 20) \]

Step 4 Solve.

\[ 10 = 0.4x + 8 \]

\[ 2 = 0.4x \]

\[ x = 5 \]

Step 5 State the answer. 5 L of water are needed.

Step 6 Check.

\[ 20(0.5) = 0.4(5 + 20) \]

\[ 10 = 0.4(25) \]

\[ 10 = 10 \]
Multiply the number of coins by the denominations, and add the results to get 8.60

<table>
<thead>
<tr>
<th>Number of Coins</th>
<th>Denominations</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>x</td>
<td>0.10</td>
<td>0.10x</td>
</tr>
<tr>
<td>26 – x</td>
<td>0.50</td>
<td>0.50(26 – x)</td>
</tr>
<tr>
<td>XXXXXXXX</td>
<td>Total</td>
<td>8.60</td>
</tr>
</tbody>
</table>

Step 3  Write an equation.

\[ 0.10x + 0.50(26 - x) = 8.60. \]
Step 4 Solve.

\[ 0.10x + 0.50(26 - x) = 8.60 \]  
Multiply by 10.

\[ 1x + 5(26 - x) = 86 \]  
Distributive property.

\[ 1x + 130 - 5x = 86 \]

\[ -4x = -44 \]

\[ x = 11 \]

Step 5 State the answer. He has 11 dimes and 26 – 11 = 15 half-dollars.

Step 6 Check. He has 11 + 15 = 26 coins, and the value is $0.10(11) + $0.50(15) = $8.60.
Objective 2

Solve problems about uniform motion.
PROBLEM-SOLVING HINT

Uniform motion problems use the distance formula, $d = rt$. In this formula, *when rate (or speed) is given in miles per hour, time must be given in hours.* To solve such problems, *draw a sketch* to illustrate what is happening in the problem, and *make a table* to summarize the given information,
EXAMPLE 2

Two cars leave the same town at the same time. One travels north at 60 mph and the other south at 45 mph. In how many hours will they be 420 mi apart?

*Step 1*  Read the problem. We are looking for the time that it takes for the cars to be 420 miles apart.

*Step 2*  Assign a variable. A sketch shows what is happening.
Let $x$ = the amount of time needed for the cars to be 420 mi apart.

<table>
<thead>
<tr>
<th></th>
<th>Rate</th>
<th>Time</th>
<th>Distance</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Northbound Car</strong></td>
<td>60</td>
<td>$x$</td>
<td>$60x$</td>
</tr>
<tr>
<td><strong>Southbound Car</strong></td>
<td>45</td>
<td>$x$</td>
<td>$45x$</td>
</tr>
<tr>
<td>XXXXXXXX</td>
<td>XXXX</td>
<td>XXXX</td>
<td>420</td>
</tr>
</tbody>
</table>

**Step 3**  
Write an equation.

$$60x + 45x = 420$$
continued

**Step 4** Solve. \(60x + 45x = 420\)

\[
105x = 420
\]

\[
x = \frac{420}{105} = 4
\]

**Step 5** State the answer. The cars will be 420 mi apart in 4 hr.

**Step 6** Check. \(60(4) + 45(4) = 420\)

\[
240 + 180 = 420
\]

\[
420 = 420
\]
**CAUTION** It is a common error to write 420 as the distance traveled by each car in Example 2. Four hundred twenty is the *total* distance traveled.

\[
\text{partial distance} + \text{partial distance} = \text{total distance}.
\]
When Michael drives his car to work, the trip takes \( \frac{1}{2} \) hr. When he rides the bus, it takes \( \frac{3}{4} \) hr. The speed of the bus is 12 mph less than the speed when driving his car. Find the distance he travels to work.

**Step 1** Read the problem. We are looking for the distance Michael travels to his workplace.

**Step 2** Assign a variable.

Let \( x \) = the speed.

Then \( x - 12 \) = the speed of the bus.
### Step 3
Write an equation.

\[
\frac{1}{2} x = \frac{3}{4} (x - 12)
\]

### Step 4
Solve.

\[
\begin{align*}
\frac{1}{2} x &= \frac{3}{4} (x - 12) \\
2x &= 3(x - 12) \\
2x &= 3x - 36 \\
36 &= x
\end{align*}
\]

Multiply by 4.
continued

Step 5  State the answer.

The required distance is

\[ d = \frac{1}{2} x = \frac{1}{2} (36) = 18 \text{ miles}. \]

Step 6  Check.

\[ d = \frac{3}{4} (x - 12) \]

\[ d = \frac{3}{4} (36 - 12) \]

\[ d = \frac{3}{4} (24) \]

\[ d = 18 \text{ miles} \]

Same result
Objective

3

Solve problems about angles.
EXAMPLE 4

Find the value of $x$, and determine the measure of each angle.

**Step 1**  Read the problem. We are asked to find the measure of each angle.

**Step 2**  Assign a variable.

Let $x = \text{the measure of one angle.}$
Step 3  Write an equation. The sum of the three measures shown in the figure must be $180^\circ$.

\[ x + (x + 61) + (2x + 7) = 180 \]

Step 4  Solve.

\[ 4x + 68 = 180 \]
\[ 4x = 112 \]
\[ x = 28 \]

Step 5  State the answer. The angles measure $28^\circ$, $28 + 61 = 89^\circ$, and $2(28) + 7 = 63^\circ$.

Step 6  Check. $28^\circ + 89^\circ + 63^\circ = 180^\circ$. 
2.5 Linear Inequalities in One Variable

1. Solve linear inequalities by using the addition property.
2. Solve linear inequalities by using the multiplication property.
3. Solve linear inequalities with three parts.
4. Solve applied problems by using linear inequalities.
## INTERVAL NOTATION

<table>
<thead>
<tr>
<th>Type of Interval</th>
<th>Set</th>
<th>Interval Notation</th>
<th>Graph</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Open interval</strong></td>
<td>{x</td>
<td>a &lt; x}</td>
<td>( (a, \infty) )</td>
</tr>
<tr>
<td></td>
<td>{x</td>
<td>a &lt; x &lt; b}</td>
<td>( (a, b) )</td>
</tr>
<tr>
<td></td>
<td>{x</td>
<td>x &lt; b}</td>
<td>( (-\infty, b) )</td>
</tr>
<tr>
<td></td>
<td>{x</td>
<td>x is a real number}</td>
<td>( (-\infty, \infty) )</td>
</tr>
<tr>
<td><strong>Half-open interval</strong></td>
<td>{x</td>
<td>a \leq x}</td>
<td>[a, \infty) ]</td>
</tr>
<tr>
<td></td>
<td>{x</td>
<td>a &lt; x \leq b}</td>
<td>( (a, b])</td>
</tr>
<tr>
<td></td>
<td>{x</td>
<td>a \leq x &lt; b}</td>
<td>[a, b) ]</td>
</tr>
<tr>
<td></td>
<td>{x</td>
<td>x \leq b}</td>
<td>( (-\infty, b])</td>
</tr>
<tr>
<td><strong>Closed interval</strong></td>
<td>{x</td>
<td>a \leq x \leq b}</td>
<td>[a, b]</td>
</tr>
</tbody>
</table>
An **inequality** says that two expressions are *not* equal.

Solving inequalities is similar to solving equations.
Linear Inequality in One Variable

A linear inequality in one variable can be written in the form

$$Ax + B < C,$$

where $A$, $B$, and $C$ are real numbers, with $A \neq 0$. 
Objective

1

Solve linear inequalities by using the addition property.
Addition Property of Inequality

For all real numbers $A$, $B$, and $C$, the inequalities

$$A < B \quad \text{and} \quad A + C < B + C$$

are equivalent.

That is, adding the same number to each side of an inequality does not change the solution set.
EXAMPLE 1

Solve \( k - 5 > 1 \) and graph the solution set.

\[
k - 5 > 1
\]

\[
k - 5 + 5 > 1 + 5 \quad \text{Add 5.}
\]

\[
k > 6
\]

**Check:** Substitute 6 for \( k \) in the equation \( k - 5 = 1 \)

\[
k - 5 = 1
\]

\[
6 - 5 = 1
\]

\[
1 = 1 \quad \text{True}
\]

This shows that 6 is a boundary point. Now test a number on each side of the 6 to verify that numbers greater than 5 make the inequality true.
Let $k = 4$

\[ 4 - 5 > 1 \]

\[ -1 > 1 \quad \text{False} \]

–1 is not in the solution set

Let $k = 7$

\[ 7 - 5 > 1 \]

\[ 2 > 1 \quad \text{True} \]

7 is in the solution set

The check confirms that $(6, \infty)$, is the correct solution.
EXAMPLE 2

Solve $5x + 3 \geq 4x - 1$ and graph the solution set.

$5x + 3 - 3 \geq 4x - 1 - 3$

$5x \geq 4x - 4$

$5x - 4x \geq 4x - 4x - 4$

$x \geq -4$

Check: $5x + 3 = 4x - 1$

$5(-4) + 3 = 4(-4) - 1$

$-20 + 3 = -16 - 1$

$-17 = -17$

This shows that $-4$ is a boundary point.
continued

5\(x + 3 \geq 4x - 1\)

Let \(x = -5\)

\[
5(-5) + 3 \geq 4(-5) - 1
\
-25 + 3 \geq -20 - 1
\
-22 \geq -21 \text{  False}
\]

-5 is not in the solution set

Let \(x = 0\)

\[
5(0) + 3 \geq 4(0) - 1
\
0 + 3 \geq 0 - 1
\
3 \geq -1 \text{  True}
\]

0 is in the solution set

The check confirms that \([-4, \infty)\), is the correct solution.
Objective 2

Solve linear inequalities by using the multiplication property.
Multiplication Property of Inequality

For all real numbers $A$, $B$, and $C$, with $C \neq 0$,

a. the inequalities

$A < B$ and $AC < BC$ are equivalent if $C > 0$;

b. the inequalities

$A < B$ and $AC > BC$ are equivalent if $C < 0$.

That is, each side of an inequality may be multiplied (or divided) by a positive number without changing the direction of the inequality symbol.

*Multiplying (or dividing) by a negative number requires that we reverse the inequality symbol.*
EXAMPLE 3

Solve each inequality and graph the solution set.

a. \( 4m \leq -100 \)

Divide each side by 4. *Since 4 > 0, do not reverse the inequality symbol.*

\[
\frac{4m}{4} \leq \frac{-100}{4}
\]

\[m \leq -25\]

The solution set is the interval \((-\infty, -25]\).
b. \(-9m < -81\)

Divide each side by \(-9\). \textit{Since \(-9 < 0\), reverse the inequality symbol.}

\[
\begin{align*}
-9m &< -81 \\
\frac{-9m}{-9} &> \frac{-81}{-9} \\
m &> 9
\end{align*}
\]

The solution set is the interval \((9, \infty)\).
Solving a Linear Inequality

**Step 1** Simplify each side separately. Use the distributive property to clear parentheses and combine like terms as needed.

**Step 2** Isolate the variable terms on one side. Use the addition property of inequality to get all terms with variables on one side of the inequality and all numbers on the other.

**Step 3** Isolate the variable. Use the multiplication property of inequality to change the inequality to the form

\[ x < k \text{ or } x > k \]
EXAMPLE 4

Solve \( 6(x - 1) + 3x \geq -x - 3(x + 2) \) and graph the solution set.

**Step 1**

\[
6(x - 1) + 3x \geq -x - 3(x + 2)
\]

\[
6x - 6 + 3x \geq -x - 3x - 6
\]

\[
9x - 6 \geq -4x - 6
\]

**Step 2**

\[
9x - 6 + 4x \geq -4x - 6 + 4x
\]

\[
13x - 6 \geq -6
\]

\[
13x - 6 + 6 \geq -6 + 6
\]

**Step 3**

\[
13x \geq 0
\]

\[
\frac{13x}{13} \geq \frac{0}{13}
\]

\[
x \geq 0
\]
EXAMPLE 5

Solve \( \frac{1}{4}(m + 3) + 2 \leq \frac{3}{4}(m + 8) \)

\[ 4 \left[ \frac{1}{4}(m + 3) + 2 \right] \leq 4 \left[ \frac{3}{4}(m + 8) \right] \]

Multiply by 4.

\[ 4 \left[ \frac{1}{4}(m + 3) \right] + 4(2) \leq 4 \left[ \frac{3}{4}(m + 8) \right] \]

Distributive property.

\[ m + 3 + 8 \leq 3(m + 8) \]

Multiply.

Distributive property.

\[ m + 3 + 8 \leq 3m + 24 \]

\[ m + 11 \leq 3m + 24 \]

Subtract 11.
continued

\[ m + 11 - 11 \leq 3m + 24 - 11 \]

Subtract 11.

\[ m \leq 3m + 13 \]

Subtract 3m.

\[ m - 3m \leq 3m - 3m + 13 \]

\[-2m \leq 13 \]

Divide \(-2\).

\[ \frac{-2m}{-2} \geq \frac{13}{-2} \]

\[ m \geq -\frac{13}{2} \]

Reverse the inequality symbol when dividing by a negative number.
Objective 3

Solve linear inequalities with three parts.
For some applications, it is necessary to work with a **three-part inequality** such as

$$3 < x + 2 < 8,$$

where $x + 2$ is between 3 and 8.
EXAMPLE 6

Solve $5 < 3x - 4 < 9$ and graph the solution set.

$5 < 3x - 4 < 9$

$5 + 4 < 3x - 4 + 4 < 9 + 4$ \hspace{1cm} \text{Add 4 to each part.}

$9 < 3x < 13$

$\frac{9}{3} < \frac{3x}{3} < \frac{13}{3}$

$3 < x < \frac{13}{3}$
**Solution Sets of Linear Equations and Inequalities**

<table>
<thead>
<tr>
<th>Equation or Inequality</th>
<th>Typical Solution Set</th>
<th>Graph of Solution Set</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linear equation</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$5x + 4 = 14$</td>
<td>${2}$</td>
<td><img src="image" alt="Graph" /></td>
</tr>
<tr>
<td>Linear inequality</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$5x + 4 &lt; 14$</td>
<td>$(-\infty, 2)$</td>
<td><img src="image" alt="Graph" /></td>
</tr>
<tr>
<td>or</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$5x + 4 &gt; 14$</td>
<td>$(2, \infty)$</td>
<td><img src="image" alt="Graph" /></td>
</tr>
<tr>
<td>Three-part inequality</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$-1 \leq 5x + 4 \leq 14$</td>
<td>$[-1, 2]$</td>
<td><img src="image" alt="Graph" /></td>
</tr>
</tbody>
</table>
Objective 4

Solve applied problems by using linear inequalities.
EXAMPLE 7

A rental company charges $5 to rent a leaf blower, plus $1.75 per hr. Marge Ruhberg can spend no more than $26 to blow leaves from her driveway and pool deck. What is the maximum amount of time she can use the rented leaf blower?

Step 1  Read the problem again.

Step 2  Assign a variable. Let \( h \) = the number of hours she can rent the leaf blower.
Step 3    Write an inequality. She must pay $5, plus $1.75 per hour for $h$ hours and no more than $26.

\[
\begin{align*}
\text{Cost of renting} & \quad \text{is no more than 26} \\
5 + 1.75h & \leq 26
\end{align*}
\]

Step 4    Solve. \hspace{1cm} 1.75h \leq 21 \\
\hspace{1cm} h \leq 12

Step 5    State the answer. She can use the leaf blower from a maximum of 12 hours.

Step 6    Check. If she uses the leaf blower for 12 hr, she will spend \(5 + 1.75(12) = 26\) dollars, the maximum.
EXAMPLE 8

Michael has scores of 92, 90, and 84 on his first three tests. What score must he make on his fourth test in order to keep an average of at least 90?

Let $x = \text{score on the fourth test}$.

His average score must be at least 90.

To find the average of four numbers, add them and then divide by 4.

\[
\frac{92 + 90 + 84 + x}{4} \geq 90
\]
He must score 94 or more on his fourth test.

\[
\frac{266 + x}{4} \geq 90
\]

\[
266 + x \geq 360
\]

\[
x \geq 94
\]

Check: \[
\frac{92 + 90 + 84 + 94}{4} \geq 90
\]

\[
\frac{92 + 90 + 84 + 94}{4} = \frac{360}{4} = 90
\]
2.6 Set Operations and Compound Inequalities

1. Find the intersection of two sets.
2. Solve compound inequalities with the word \textit{and}.
3. Find the union of two sets.
4. Solve compound inequalities with the word \textit{or}.
Objective 1

Find the intersection of two sets.
Intersection of Sets

For any two sets $A$ and $B$, the **intersection** of $A$ and $B$, symbolized $A \cap B$, is defined as follows:

$$A \cap B = \{ x \mid x \text{ is an element of } A \text{ and } x \text{ is an element of } B \}.$$
EXAMPLE 1

Let $A = \{3, 4, 5, 6\}$ and $B = \{5, 6, 7\}$. Find $A \cap B$.

The set $A \cap B$, the intersection of $A$ and $B$, contains those elements that belong to both $A$ and $B$; that is, the numbers 5 and 6.

$A \cap B = \{3, 4, 5, 6\} \cap \{5, 6, 7\}$

Therefore,

$A \cap B = \{5, 6\}$.
A compound inequality consists of two inequalities linked by a consecutive word such as \textit{and} or \textit{or}.

Examples of compound inequalities are

\begin{align*}
x + 1 &\leq 9 \quad \text{and} \quad x - 2 &\geq 3 \\
2x &> 4 \quad \text{or} \quad 3x - 6 &< 5.
\end{align*}
Objective 2

Solve compound inequalities with the word *and*. 
Solving a Compound Inequality with *and*

*Step 1*  Solve each inequality individually.

*Step 2*  Since the inequalities are joined with and, the solution set of the compound inequality will include all numbers that satisfy both inequalities in Step 1 (the intersection of the solution sets).
EXAMPLE 2

Solve the compound inequality and graph the solution set. \( x + 3 < 1 \) and \( x - 4 > -12 \)

**Step 1** Solve each inequality individually.

\[
x + 3 < 1 \quad \text{and} \quad x - 4 > -12
\]

\[
x + 3 - 3 < 1 - 3 \quad \text{and} \quad x - 4 + 4 > -12 + 4
\]

\[
x < -2 \quad \text{and} \quad x > -8
\]
Step 2  Because the inequalities are joined with the word and, the solution set will include all numbers that satisfy both inequalities.

The solution set is \((-8, -2)\).
EXAMPLE 3

Solve the compound inequality and graph the solution set.

\[ 2x \leq 4x + 8 \quad \text{and} \quad 3x \geq -9 \]

**Step 1** Solve each inequality individually.

\[ 2x \leq 4x + 8 \quad \text{and} \quad 3x \geq -9 \]

\[ -2x \leq 8 \quad \quad \quad \quad x \geq -3 \]

\[ x \geq -4 \]

*Remember to reverse the inequality symbol.*
continued

Step 2 The overlap of the graphs consists of the numbers that are greater than or equal to \(-4\) and are also greater than or equal to \(-3\).

The solution set is \([-3, \infty)\).
EXAMPLE 4

Solve $x + 2 > 3$ and $2x + 1 < -3$.

Solve each inequality individually.

$x + 2 > 3$ and $2x + 1 < -3$

$x > 1$ and $2x < -4$

$x > 1$ and $x < -2$
There is no number that is both greater than 1 and less than \(-2\), so the given compound inequality has no solution.

The solution set is \(\emptyset\).
Objective 3

Find the union of two sets.
Union of Sets

For any two sets $A$ and $B$, the union of $A$ and $B$, symbolized $A \cup B$, is defined as follows:

$$A \cup B = \{ x \mid x \text{ is an element of } A \text{ or } x \text{ is an element of } B \}.$$
EXAMPLE 5

Let \( A = \{3, 4, 5, 6\} \) and \( B = \{5, 6, 7\} \). Find \( A \cup B \).

The set \( A \cup B \), the intersection of \( A \) and \( B \), consists of all elements in either \( A \) or \( B \) (or both). Start by listing the elements of set \( A \): 3, 4, 5, 6. Then list any additional elements from set \( B \). In this case, the elements 5 and 6 are already listed, so the only additional element is 7.

Therefore,

\[
A \cup B = \{3, 4, 5, 6, 7\}.
\]
Objective

Solve compound inequalities with the word *or*. 
**Solving a Compound Inequality with or**

**Step 1**  Solve each inequality individually.

**Step 2**  Since the inequalities are joined with *or*, the solution set of the compound inequality includes all numbers that satisfy either one of the two inequalities in Step 1 (the union of the solution sets).
EXAMPLE 6

Solve \( x - 1 > 2 \) or \( 3x + 5 < 2x + 6 \).

**Step 1**  Solve each inequality individually.

\[
x - 1 > 2 \quad \text{or} \quad 3x + 5 < 2x + 6
\]

\[x > 3\] \[x < 1\]
The graph of the solution set consists of all numbers greater than 3 or less than 1.

The solution set is \((-\infty, 1) \cup (3, \infty)\).
EXAMPLE 7

Solve $3x - 2 \leq 13$ or $x + 5 \leq 7$.

Solve each inequality individually.

$3x - 2 \leq 13$ or $x + 5 \leq 7$

$3x \leq 15$

$x \leq 5$ or $x \leq 2$

$x \leq 5$

$x \leq 2$
The solution set is all numbers that are either less than or equal to 5 or less than or equal to 2. All real numbers less than or equal to 5 are included.

The solution set is \((-\infty, 5]\).
EXAMPLE 8

Solve \( 3x - 2 \leq 13 \) or \( x + 5 \geq 7 \).

Solve each inequality individually.

\[
3x - 2 \leq 13 \quad \text{or} \quad x + 5 \geq 7
\]

\[
3x \leq 15 \quad \text{or} \quad x \geq 2
\]

\[
x \leq 5 \quad \text{or} \quad x \geq 2
\]
The solution set is all numbers that are either less than or equal to 5 or greater than or equal to 2. All real numbers are included.

The solution set is \((-\infty, -\infty)\).
EXAMPLE 9

The five highest grossing domestic films as of July 2005 are listed in this table. List the elements that satisfy each set.

<table>
<thead>
<tr>
<th>Film</th>
<th>Admissions</th>
<th>Gross Income</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gone with the Wind</td>
<td>202,044,569</td>
<td>$1,293,085,000</td>
</tr>
<tr>
<td>Star Wars</td>
<td>178,119,595</td>
<td>$1,139,965,000</td>
</tr>
<tr>
<td>The Sound of Music</td>
<td>142,415,376</td>
<td>$911,458,000</td>
</tr>
<tr>
<td>E.T.</td>
<td>141,925,359</td>
<td>$908,322,298</td>
</tr>
<tr>
<td>The Ten Commandments</td>
<td>131,000,000</td>
<td>$838,400,000</td>
</tr>
</tbody>
</table>

Source: Exhibitor Relations Co., Inc.
a. The set of films with admissions greater than 130,000,000 and gross income less than $800,000,000.

b. The set of films with admissions greater than 130,000,000 or gross income less than $500,000,000.

Solution

a. All films had admissions greater than 130,000,000, but no films had a gross income of less than $800,000,000. Thus, there are no elements (films) that satisfy both conditions, so the required set is the empty set, symbolized by $\emptyset$.

b. Since all the films had admissions greater than 130,000,000 and we have an or statement, the second condition doesn’t have an effect on the solution set. The required set is the set of all films that is $\{Gone\ with\ the\ Wind,\ Stars\ Wars,\ The\ Sound\ of\ Music,\ E.T.,\ The\ Ten\ Commandments\}$. 
2.7 Absolute Value Equations and Inequalities

1. Use the distance definition of absolute value.
2. Solve equations of the form $|ax + b| = k$, for $k > 0$.
3. Solve inequalities of the form $|ax + b| < k$ and of the form $|ax + b| > k$, for $k > 0$.
4. Solve absolute value equations that involve rewriting.
5. Solve equations of the form $|ax + b| = |cx + d|$.
6. Solve special cases of absolute value equations and inequalities.
Objective

1

Use the distance definition of absolute value.
The **absolute value** of a number \( x \), written \( |x| \), is the distance from \( x \) to 0 on the number line.

For example, the solutions of \( |x| = 5 \) are 5 and \(-5\), as shown below.

Distance is 5, so \(|-5| = 5\).  
Distance is 5, so \(|5| = 5\).
**Solving Absolute Value Equations and Inequalities**

Let $k$ be a positive real number and $p$ and $q$ be real numbers.

1. To solve $|ax + b| = k$, solve the compound equation
   
   $$ax + b = k \quad \text{or} \quad ax + b = -k.$$

   The solution set is usually of the form \{p, q\}, which includes two numbers.

   ![Graph showing two points, p and q, and the interval (p, q).]

2. To solve $|ax + b| > k$, solve the compound inequality
   
   $$ax + b > k \quad \text{or} \quad ax + b < -k.$$

   The solution set is of the form $(-\infty, p) \cup (q, \infty)$, which consists of two separate intervals.

   ![Graph showing two intervals, p to the left and q to the right, and the open intervals (p, q).]

3. To solve $|ax + b| < k$, solve the three-part inequality
   
   $$-k < ax + b < k.$$

   The solution set is of the form $(p, q)$, a single interval.

   ![Graph showing a single interval, p to the left and q to the right.]
Objective 2

Solve equations of the form $|ax + b| = k$, for $k > 0$. 
EXAMPLE 1

Solve $|3x - 4| = 11$.

$$3x - 4 = -11 \quad \text{or} \quad 3x - 4 = 11$$

$$3x - 4 + 4 = -11 + 4 \quad \quad 3x - 4 + 4 = 11 + 4$$

$$3x = -7 \quad \quad 3x = 15$$

$$x = -7/3 \quad \quad x = 5$$

Check by substituting $-7/3$ and $5$ into the original absolute value equation to verify that the solution set is $\{-7/3, 5\}$. 
Objective 3

Solve inequalities of the form $|ax + b| < k$ and of the form $|ax + b| > k$, for $k > 0$. 
EXAMPLE 2

Solve $|3x - 4| \geq 11$.

$3x - 4 \leq -11$ or $3x - 4 \geq 11$

$3x - 4 + 4 \leq -11 + 4$ or $3x - 4 + 4 \geq 11 + 4$

$3x \leq -7$ or $3x \geq 15$

$x \leq -7/3$ or $x \geq 5$

Check the solution. The solution set is $(-\infty, -7/3] \cup [5, \infty)$. The graph consists of two intervals.
EXAMPLE 3

Solve $|3x - 4| < 11$.

$-11 < 3x - 4 < 11$

$-11 + 4 < 3x - 4 < 11 + 4$

$-7 < 3x < 15$

$-7/3 < x < 5$

Check the solution. The solution set is $(-7/3, 5)$. The graph consists of a single interval.
Objective 4

Solve absolute value equations that involve rewriting.
EXAMPLE 4

Solve $|3a + 2| + 4 = 15$.
First get the absolute value alone on one side of the equals sign.

$$|3a + 2| + 4 = 15$$
$$|3a + 2| + 4 - 4 = 15 - 4$$
$$|3a + 2| = 11$$
$$3a + 2 = -11 \quad \text{or} \quad 3a + 2 = 11$$
$$3a = -13 \quad \quad 3a = 9$$
$$a = -13/3 \quad \quad a = 3$$

Check that the solution set is $\{-13/3, 3\}$. 
EXAMPLE 5

Solve each inequality.

a. $|x + 2| - 3 > 2$.

$|x + 2| - 3 = 2$

$|x + 2| = 5$

$x + 2 > 5$  or  $x + 2 < -5$

$x > 3$  $x < -7$

Solution set: $(-\infty, -7) \cup (3, \infty)$. 
b. \[ |3x + 2| + 4 \leq 15. \]

\[ |3x + 2| + 4 \leq 15 \]
\[ |3x + 2| \leq 11 \]
\[ -11 \leq 3x + 2 \leq 11 \]
\[ -13 \leq 3x \leq 9 \]
\[ -13/3 \leq x \leq 3 \]

Solution set: \[ \left[ -\frac{13}{3}, 3 \right] \]
Objective 5

Solve equations of the form $|ax + b| = |cx + d|$.
Solving $|ax + b| = |cx + d|$

To solve an absolute value equation of the form $|ax + b| = |cx + d|$, solve the compound equation

$$ax + b = cx + d \quad \text{or} \quad ax + b = -(cx + d).$$
EXAMPLE 5

Solve $|4r - 1| = |3r + 5|$.

$4r - 1 = 3r + 5$ or $4r - 1 = -(3r + 5)$

$4r - 6 = 3r$ or $4r - 1 = -3r - 5$

$-6 = -r$ or $7r = -4$

$r = 6$ or $r = -4/7$

Check that the solution set is $\left\{-\frac{4}{7}, 6\right\}$. 
Objective

Solve special cases of absolute value equations and inequalities.
Special Cases of Absolute Value

1. The absolute value of an expression can never be negative; that is, \( |a| \geq 0 \) for all real numbers \( a \).

2. The absolute value of an expression equals 0 only when the expression is equal to 0.
EXAMPLE 6

Solve each equation.

a. \[ |6x + 7| = -5 \]

\[ |6x + 7| = -5 \]

*The absolute value of an expression can never be negative, so there are no solutions for this equation.*

The solution set is \( \emptyset \).
b. \( \left[ \frac{1}{4} x - 3 \right] = 0 \)

The expression \( \frac{1}{4} x - 3 \) will equal 0 *only* if

\[
\frac{1}{4} x - 3 = 0
\]

The solution of the equation is 12.
The solution set is \( \{12\} \), with just one element.
EXAMPLE 7

Solve each inequality.

a. \(|x| > -5\)

\textit{The absolute value of a number is always greater than or equal to 0.}

The solution set is \((&-\infty, \infty&)\).

b. \(|t - 10| - 2 \leq -3\)

\[|t - 10| \leq -1\] Add 2 to each side.

There is no number whose absolute value is less than \(-1\), so the inequality has no solution.

The solution set is \(\emptyset\).
continued

c. $|x + 2| \leq 0$

The value of $|x + 2|$ will never be less than 0. $|x + 2|$ will equal 0 when $x = -2$. The solution set is $\{-2\}$. 