Math 080

Beginning Algebra
Chapter 1

Variables, Real Numbers and Mathematical Models
§ 1.1

Introduction to Algebra: Variables and Mathematical Models
Algebra uses letters such as $x$ and $y$ to represent numbers. If a letter is used to represent various numbers, it is called a variable. For example, the variable $x$ might represent the number of minutes you can lie in the sun without burning when you are not wearing sunscreen.
Suppose you are wearing number 6 sunscreen. If you can normally lie in the sun \( x \) minutes without burning, with the number 6 sunscreen, you can lie in the sun 6 times as long without burning - that is, 6 times \( x \) or \( 6x \) would represent your exposure time without burning.
A combination of variables and numbers using the operations of addition, subtraction, multiplication, or division, as well as powers or roots, is called an algebraic expression.
## Translating Phrases into Expressions

<table>
<thead>
<tr>
<th>English Phrase</th>
<th>Mathematical Operation</th>
</tr>
</thead>
<tbody>
<tr>
<td>sum</td>
<td>Addition</td>
</tr>
<tr>
<td>plus</td>
<td></td>
</tr>
<tr>
<td>increased by</td>
<td></td>
</tr>
<tr>
<td>more than</td>
<td></td>
</tr>
<tr>
<td>difference</td>
<td>Subtraction</td>
</tr>
<tr>
<td>minus</td>
<td></td>
</tr>
<tr>
<td>decreased by</td>
<td></td>
</tr>
<tr>
<td>less than</td>
<td></td>
</tr>
<tr>
<td>product</td>
<td>Multiplication</td>
</tr>
<tr>
<td>times</td>
<td></td>
</tr>
<tr>
<td>of (used with fractions)</td>
<td></td>
</tr>
<tr>
<td>twice</td>
<td></td>
</tr>
<tr>
<td>quotient</td>
<td>Division</td>
</tr>
<tr>
<td>divide</td>
<td></td>
</tr>
<tr>
<td>per</td>
<td></td>
</tr>
<tr>
<td>ratio</td>
<td></td>
</tr>
</tbody>
</table>
EXAMPLE

Write the English phrase as an algebraic expression. Let $x$ represent the number.

Four more than five times a number

SOLUTION

$5x + 4$
Evaluating an Algebraic Expression

**EXAMPLE**

Evaluate each algebraic expression for $x = 2$.

a.  $5 + 3x$

b.  $5(x + 7)$

**SOLUTION**

Problem (a).

$5 + 3(2)$  Replace the $x$ with 2.
$5 + 6$       Perform the multiplication
$11$         Perform the addition

Problem (b).

$5(2 + 7)$  Replace the $x$ with 2.
$5(9)$      Perform the addition.
$45$       Perform the multiplication.
In evaluating expressions, what comes first?

1. Start with the parentheses. Parentheses say “Me First!”
2. Then evaluate the exponential expressions.
3. Multiplications and divisions are equal in the order of operations – Perform them next.
4. Additions and subtractions are also equal to each other in order – and they come last.

Remember by “PEMDAS” - parentheses, exponents, multiplication, division, addition, subtraction

Blitzer, Introductory Algebra, 5e – Slide #10 Section 1.1
### Order of Operations - PEMDAS

1) First, perform *all* operations within grouping symbols.

2) Next, Evaluate *all* exponential expressions.

3) Next, do all multiplications and divisions in the order in which they occur working from *left to right*.

4) Finally, do all additions and subtractions in the order in which they occur, working from *left to right*. 
Order of Operations

EXAMPLE

Simplify. \( 6 + \frac{4 \cdot 3^2}{6} - 2 \)

SOLUTION

\[
6 + \frac{4 \cdot 3^2}{6} - 2 \\
6 + \frac{4 \cdot 9}{6} - 2 \\
6 + \frac{36}{6} - 2 \\
6 + 6 - 2 \\
-2
\]

Evaluating exponent

Multiply

Divide

Subtract
Order of Operations - PEMDAS

EXAMPLE

Evaluate $R^3 - 2 \, R \, 3$ for $R = 3$.

SOLUTION

$R^3 - 2 \, R \, 3$

$3^3 - 2 \, 3 \, 3$

Replace $R$ with 3

$3^3 - 2 \, 3^3$

Evaluate inside parentheses first

$27 - 2 \, 27$

Evaluate $3^3$ – first exponent
<table>
<thead>
<tr>
<th>Expression</th>
<th>Operation</th>
</tr>
</thead>
<tbody>
<tr>
<td>27 - 2 \times 81</td>
<td>Evaluate $3^4$ – second exponent</td>
</tr>
<tr>
<td>27 - 162</td>
<td>Multiply</td>
</tr>
<tr>
<td>-135</td>
<td>Subtract</td>
</tr>
</tbody>
</table>
An equation is a statement that two algebraic expressions are equal. An equation always contains the equality symbol $=$. Some examples of equations are:

\[ 5x - 2 = 15 \quad 3x + 7 = 2x \quad 3(z - 1) = 4(z + 7) \]
Solutions of equations are values of the variable that make the equation a true statement. To determine whether a number is a solution, substitute that number for the variable and evaluate both sides of the equation. If the values on both sides of the equation are the same, the number is a solution.

For example, 2 is a solution of \( x + 4 = 3x \) since when we substitute the 2 for \( x \), we get \( 2 + 4 = 3(2) \) or equivalently, \( 6 = 6 \).
One aim of algebra is to provide a compact, symbolic description of the world. A formula is an equation that expresses a relationship between two or more variables.

One variety of crickets chirps faster as the temperature rises. You can calculate the temperature by applying the following formula:

\[ T = 0.3n + 40 \]

If you are sitting on your porch and hear 50 chirps per minute, then the temperature is:

\[ T = 0.3(50) + 40 = 15 + 40 = 55 \text{ degrees} \]
The process of finding formulas to describe real-world phenomena is called **mathematical modeling**. Formulas together with the meaning assigned to the variables are called mathematical models.

In creating mathematical models, we strive for both *simplicity* and *accuracy*. For example, the cricket formula is relatively easy to use. But you should not get upset if you count 50 chirps per minute and the temperature is 53 degrees rather than 55. Many mathematical formulas give an *approximate rather than exact* description of the relationship between variables.
§ 1.2

Fractions in Algebra
Example:

The number above the fraction bar is the **numerator** and the number below the fraction bar is the **denominator**.
Example:

This fraction is read:
“five sevenths”.

The meaning is: 5 of 7 equal parts of a whole.
Natural Numbers

• The numerators and denominators of the fractions we will be working with are *natural numbers*.

• The natural numbers are the numbers we use for counting:

  1, 2, 3, 4, 5, ...

  The three dots after the 5 indicate that the list continues in the same manner without ending.
Mixed Numbers

- A mixed number consists of the addition of a natural number and a fraction, expressed without the addition symbol. In this example, the natural number is 2 and the fraction is $\frac{3}{4}$.
Converting a Mixed Number to an Improper Fraction

1. Multiply the denominator of the fraction by the natural number and add the numerator to this product.

2. Place the result from step 1 over the denominator in the mixed number.
Convert $2 \frac{3}{4}$ to an improper fraction.

\[
2 \frac{3}{4} = \frac{4 \cdot 2 + 3}{4} = \frac{11}{4}
\]

Mixed number $\rightarrow$ Improper fraction
Converting an improper fraction to a mixed number

1. Divide the denominator into the numerator. Record the quotient and the remainder.

2. Write the mixed number using the form:

\[
\text{quotient} \quad \frac{\text{remainder}}{\text{original denominator}}.
\]
Convert $15/2$ to a mixed number.

\[
\frac{15}{2} = 7 \frac{1}{2}
\]

15 divided by 2 yields 7 with a remainder of 1

Improper fraction $\rightarrow$ Mixed number
Prime Numbers

• A prime number is a natural number greater than 1 that has only itself and 1 as factors.

• The first 10 prime numbers are 2, 3, 5, 7, 11, 13, 17, 19, 23 and 29

Note: 2 is the only even number that is prime.
Composite Numbers

• A composite number is a natural number greater than 1 that is not prime.

• Every composite number can be expressed as the product of prime numbers.

• For example, 90 is a composite number that can be expressed as

\[ 90 = 2 \cdot 3 \cdot 3 \cdot 5 \]
Fundamental Principle of Fractions

If \( \frac{a}{b} \) is a fraction and \( c \) is a nonzero number, then

\[
\frac{a \cdot c}{b \cdot c} = \frac{a}{b}.
\]
Reducing a Fraction to its Lowest Terms

1. Write the prime factorization of the numerator and denominator.

2. Divide the numerator and denominator by the greatest common factor, which is the product of all factors common to both.

( In other words, you may cancel common factors. That is why when reducing fractions you must factor numerator and denominator first)
## Writing Fractions in Lowest Terms

1) Factor the numerator and the denominator completely.

2) Divide both the numerator and the denominator by any common factors.
Reduce the fraction $\frac{15}{27}$ to its lowest terms.

\[
\frac{15}{27} = \frac{3 \cdot 5}{3 \cdot 3 \cdot 3} = \frac{5}{3 \cdot 3} = \frac{5}{9}
\]
Multiplying Fractions

If \( \frac{a}{b} \) and \( \frac{c}{d} \) are fractions, then

\[
\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}.
\]
## Multiplying Fractions

1. Factor all numerators and denominators completely.
2. Divide numerators and denominators by common factors.
3. Multiply the remaining factors in the numerators and multiply the remaining factors in the denominators.
Multiply fractions.

Multiply \( \frac{2}{3} \cdot \frac{5}{6} \).

\[
\frac{2 \cdot 5}{3 \cdot 6} = \frac{10}{18} = \frac{5}{9}
\]
Dividing Fractions

If \( \frac{a}{b} \) and \( \frac{c}{d} \) are fractions, then

\[
\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{a \cdot d}{b \cdot c}.
\]
You should memorize the definition of division for fractions.

\[
\frac{P}{Q} \div \frac{R}{S} = \frac{P}{Q} \cdot \frac{S}{R} = \frac{PS}{QR}.
\]

- Change division to multiplication.
- Replace \( \frac{R}{S} \) with its reciprocal by interchanging its numerator and denominator.
Adding Fractions

In this section, you will also practice adding and subtracting fractions. When adding or subtracting fractions, it is necessary to rewrite the fractions as fractions having the same denominator, which is called the common denominator for the fractions being combined.
### Adding Fractions With Common Denominators

To add fractions with the same denominator, add numerators and place the sum over the common denominator. If possible, simplify the result.

\[
\frac{P}{R} + \frac{Q}{R} = \frac{P+Q}{R}
\]
Subtracting Fractions With Common Denominators

\[
\frac{P}{R} - \frac{Q}{R} = \frac{P - Q}{R}
\]

To subtract fractions with the same denominator, subtract numerators and place the difference over the common denominator. If possible, simplify the result.
Adding and Subtracting Fractions with Identical Denominators

\[
\frac{a}{b} + \frac{c}{b} = \frac{a + c}{b}
\]

\[
\frac{a}{b} - \frac{c}{b} = \frac{a - c}{b}
\]
Add or subtract fractions with same denominator.

Add:
\[
\frac{2}{3} + \frac{4}{3} = \frac{2 + 4}{3} = \frac{6}{3} = 2
\]

Subtract:
\[
\frac{5}{7} - \frac{2}{7} = \frac{5 - 2}{7} = \frac{3}{7}
\]
Adding or Subtracting Fractions having *Unlike* Denominators

1. Rewrite each fractions as an equivalent fraction having the least common denominator.
2. Now you may add or subtract the numerators, putting this result over the common denominator.
### Finding the Least Common Denominator (LCD)

1. Factor each denominator completely.
2. List the factors of the first denominator.
3. Add to the list in step 2 any factors of the second denominator that do not appear in the list.
4. Form the product of each different factor from the list in step 3. This product is the least common denominator.
Add or subtract fractions having unlike denominators.

\[
\frac{2}{15} + \frac{5}{9}
\]

Find the lowest common denominator by using the prime factorization of each denominator.

\[15 = 3 \cdot 5\]
\[9 = 3 \cdot 3\]
### Adding and Subtracting Fractions That Have Different Denominators

1. Find the LCD of the fractions.

2. Rewrite each fraction as an equivalent fractions whose denominator is the LCD. To do so, multiply the numerator and denominator of each fraction by any factor(s) needed to convert the denominator into the LCD.

3. Add or subtract numerators, placing the resulting expression over the LCD.

4. If possible, simplify the resulting fraction.
The least common denominator is found by using each factor the greatest number of times it appears in any denominator. The common denominator here is: \(5 \times 3 \times 3 = 45\)

\[
\frac{2}{15} \cdot \frac{3}{3} + \frac{5}{9} \cdot \frac{5}{5} = \frac{6}{45} + \frac{25}{45} = \frac{31}{45}
\]
Important to Remember

Adding or Subtracting Fractions

If the denominators are the same, add or subtract the numerators and place the result over the common denominator.

If the denominators are different, write all fractions with the least common denominator (LCD). Once all fractions are written in terms of the LCD, then add or subtract as described above.

In either case, simplify the result, if possible. Even when you have used the LCD, it may be true that the sum of the fractions can be reduced.
Addition of Fractions

Important to Remember

Finding the Least Common Denominator (LCD)

The LCD is consists of the product of all prime factors in the denominators, with each factor raised to the greatest power of its occurrence in any denominator.

That is - After factoring the denominators completely, the LCD can be determined by taking each factor to the highest power it appears in any factorization.

The *Mathematics Teacher* magazine accused the LCD of trying to keep up with the Joneses. The LCD wants everything (all of the factors) the other denominators have.
§ 1.3

The Real Numbers
In this section, we will look at some number sets. Before we do that, we should consider the idea of a **set**.

A **set** is a collection of objects whose contents can be clearly determined. The objects in the set are called the **elements** of the set.

The set of numbers used for counting can be represented by:

\{1,2,3,4,5,...\}

The braces, \{\}, indicate that we are representing a set. This form of representing a set uses commas to separate the elements of the set.
The set of counting numbers is also called the set of natural numbers. That set is: \{1,2,3,4,5,\ldots\}

When we extend that set to include 0, we have the set of whole numbers: \{0,1,2,3,4,5,\ldots\}

However, there are some everyday situations that we cannot describe using just these two number sets. Have you ever known the temperature to drop below 0? Or have you ever overdrawn your checking account? We have need of negative numbers also.
Number Sets

We now consider the set of **integers**. That set contains negative numbers as well as positive ones and also contains 0. The set of integers is:  {..., -5,-4,-3,-2,-1,0,1,2,3,4,5,...}

Write a negative integer that describes each of the following situations:

a. You owe a debt of $34.
b. The land is 100 feet below sea level.
c. The temperature dropped 10 degrees below 0.

Answers:

a. A debt of $34 can be expressed by the negative integer -34.
b. The land is 100 feet below sea level of 0, or is at -100.
c. The temperature is at -10 degrees.
The **number line** is the graph we use to visualize the set of integers as well as other sets of numbers. The number line extends indefinitely in both directions. Zero separates the positive numbers from the negative numbers on the number line. The positive integers are located to the right of 0 and the negative integers are located to the left of 0. *Zero is neither positive nor negative.*

For every positive integer on a number line, there is a corresponding negative integer on the opposite side of 0.
### Number Sets

<table>
<thead>
<tr>
<th>Sets of Numbers</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>Natural Numbers</td>
<td>All numbers in the set {1,2,3,4,…}</td>
</tr>
<tr>
<td>Whole Numbers</td>
<td>All numbers in the set {0,1,2,3,4,…}</td>
</tr>
<tr>
<td>Integers</td>
<td>All numbers in the set {…-3,-2,-1,0,1,2,3,…}</td>
</tr>
<tr>
<td>Rational Numbers</td>
<td>All numbers (a/b) such that (a) and (b) are integers</td>
</tr>
<tr>
<td>Irrational Numbers</td>
<td>All numbers whose decimal representation neither terminate nor repeat</td>
</tr>
<tr>
<td>Real Numbers</td>
<td>All numbers that are rational or irrational</td>
</tr>
</tbody>
</table>

Remember that : “…” means to continue without end
Three Common Number Sets

Note that...

The natural numbers are the numbers we use for counting.

The set of whole numbers includes the natural numbers and 0. Zero is a whole number, but is not a natural number.

The set of integers includes all the whole numbers and their negatives. Every whole number is an integer, and every natural number is an integer.

These sets are just getting bigger and bigger...
Rational Numbers

The set of rational numbers is the set of all numbers that can be expressed in the form $a/b$ where $a$ and $b$ are integers and $b$ is not equal to zero. In decimal form, each rational number will terminate or will repeat in a block.

Each of the following is a rational number:

$$\frac{3}{4}, \quad .333..., \quad .25$$
Rational Numbers

Definition

The set of *rational numbers* is the set of all numbers that can be expressed as the quotient of two integers with the denominator not zero.

That is, a rational number is any number that can be written in the form \( \frac{a}{b} \) where \( a \) and \( b \) are integers and \( b \) is not zero.

Rational numbers can be expressed either in fraction or in decimal notation. *Every integer is rational* because it can be written in terms of division by one.
• **Irrational Numbers**
  
  – The set of irrational numbers is the set of all numbers whose decimal representations are neither terminating nor repeating. Irrational numbers cannot be expressed as a quotient of integers.

  *Each of the following three numbers is an irrational number.*

  \[
  \pi, \sqrt{3}, -\sqrt{2}
  \]
All numbers that can be represented by points on the number line are called **real numbers**.

The set of real numbers is formed by combining the rational numbers and the irrational numbers, thus we can say that the set of real numbers is the union of the rationals and the irrationals.

Every real number is either rational or irrational, and every real number has a home on the number line, whether that home is labeled or not – it is there.
Now... let’s look again at the number sets.

<table>
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<tr>
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<td>Natural Numbers</td>
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<tr>
<td>Whole Numbers</td>
<td>All numbers in the set {0,1,2,3,4,…}</td>
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<tr>
<td>Integers</td>
<td>All numbers in the set {…-3,-2,-1,0,1,2,3,…}</td>
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<tr>
<td>Rational Numbers</td>
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<tr>
<td>Irrational Numbers</td>
<td>All numbers whose decimal representation neither terminate nor repeat</td>
</tr>
<tr>
<td>Real Numbers</td>
<td>All numbers that are rational or irrational</td>
</tr>
</tbody>
</table>

You should think about these sets and their names and try to remember them - for we will frequently refer to the sets by name.
### Ordering the Real Numbers

<table>
<thead>
<tr>
<th>Inequalities</th>
<th>Meanings</th>
<th>Examples</th>
</tr>
</thead>
</table>
| <            | is less than | 10 < 32  
|              |           | -5 < 3   
|              |           | -7 < -2  |
| >            | is greater than | 6 > -4  
|              |           | 11 > 8   
|              |           | -6 > -12 |
| ≤            | is less than or is equal to | 3.4 ≤ 4.5 
|              |           | -2 ≤ -2  |
| ≥            | is greater than or is equal to | 5 ≥ 5    
|              |           | 0 ≥ -3   |
Finding Absolute Value

Absolute value is used to describe how to operate with positive and negative numbers.

**Geometric Meaning of Absolute Value**

The *absolute value* of a real number $a$, denoted $|a|$, is the distance from 0 to $a$ on the number line. This distance is always nonnegative.

$|-5| = +5$  
The absolute value of -5 is 5 because -5 is 5 units from 0 on the number line.

$|3| = +3$  
The absolute value of 3 is +3 because 3 is 3 units from 0 on the number line.
§ 1.4

Basic Rules of Algebra
Algebraic Expression

• An algebraic expression is a combination of variables and numbers using the operations of addition, subtraction, multiplication, or division, as well as powers or roots.
  – A variable is a letter such as $x$ or $y$ used to represent numbers.
• Examples of algebraic expressions:

\[2x + 3, \sqrt{3x} - 7, 2x - 15, \frac{5x}{8}\]
Evaluating Algebraic Expressions

• If we replace the variable in an algebraic expression by a number, we are evaluating the expression.

• For example:
  – If the algebraic expression is $2x + 5$, and we evaluate it at $x = 3$, the expression then becomes $2(3) + 5 = 13$. 
Vocabulary of Algebraic Expressions

- **Terms:** The *terms* of an algebraic expression are those parts that are separated by addition.
  - In the expression $2x + 8$, $2x$ is a term and $8$ is a term.
- **Coefficient:** The numerical part of a term.
  - In the term $2x$, $2$ is the coefficient.
- **Like terms:** *Like terms* have exactly the same variable part.
  - $5x$ and $8x$ are like terms.
  - $2y$ and $7y$ are like terms.
  - $2x$ and $3y$ are not like terms.
Commutative Property

- Let $a$, $b$, and $c$ represent real numbers, variables, or algebraic expressions.

  - **Commutative Property of Addition** $a + b = b + a$
    Changing order in addition does not affect the sum.

  - **Commutative Property of Multiplication** $ab = ba$
    Changing order in multiplication does not affect the product.
Associative Property

• Let $a$, $b$, and $c$ represent real numbers, variables or algebraic expressions.

– **Associative Property of Addition**
  
  \[(a + b) + c = a + (b + c)\]

  *Changing grouping when adding does not affect the sum.*

– **Associative Property of Multiplication**
  
  \[(ab)c = a(bc)\]

  *Changing grouping when multiplying does not affect the product.*
Using the Commutative and Associative Properties

Simplify: \(8 + (3 + x)\)

\((8 + 3) + x\) \quad \text{Associative Property of Addition}

\(11 + x\) \quad \text{Perform the addition}

Simplify: \(7 + (x + 5)\)

\(7 + (5 + x)\) \quad \text{Commutative Property of Addition}

\((7 + 5) + x\) \quad \text{Associative Property of Addition}

\(12 + x\) \quad \text{Perform the addition}
Distributive Property

• Let $a$, $b$, and $c$ represent real number, variables or algebraic expressions.

$$a(b + c) = ab + ac$$

Multiplication distributes over addition.

• Example:

$$7(x + 3) = 7x + 7 \cdot 3$$

$$= 7x + 21$$
Basic Algebraic Properties

<table>
<thead>
<tr>
<th>Property</th>
<th>Examples</th>
</tr>
</thead>
<tbody>
<tr>
<td>Commutative</td>
<td>$2 + 3 = 3 + 2$</td>
</tr>
<tr>
<td></td>
<td>$10 + 4 = 4 + 10$</td>
</tr>
<tr>
<td></td>
<td>$8 + 7 = 7 + 8$</td>
</tr>
<tr>
<td></td>
<td>$2(3) = 3(2)$</td>
</tr>
<tr>
<td></td>
<td>$4(10) = 10(4)$</td>
</tr>
<tr>
<td></td>
<td>$7(8) = 8(7)$</td>
</tr>
<tr>
<td>Associative</td>
<td>$4 + (3 + 2) = (4 + 3) + 2$</td>
</tr>
<tr>
<td></td>
<td>$(6 \cdot 4)11 = 6(4 \cdot 11)$</td>
</tr>
<tr>
<td></td>
<td>$3(2 \cdot 5) = (3 \cdot 2)5$</td>
</tr>
<tr>
<td>Distributive</td>
<td>$7(2x + 3) = 14x + 21$</td>
</tr>
<tr>
<td></td>
<td>$5(3x-2-4y) = 15x - 10 - 20y$</td>
</tr>
<tr>
<td></td>
<td>$(2x + 7)4 = 8x + 28$</td>
</tr>
</tbody>
</table>
Combining Like Terms

- Add or subtract the coefficients of the terms.

Example:

\[7x + 3x\]

The terms are *like terms*, therefore to combine the terms, add the coefficients, \(7+3=10\). This is the coefficient of the combined term.

\[7x + 3x = 10x\]
Combining Like Terms

Simplify: $2x + 8 + 5x + 7$
$(2x + 5x) + (8 + 7)$  Rearrange terms and group like terms using the commutative and associative properties.
$7x + 15$  Combine like terms.

Simplify: $3y + 2x + 4x + 7y$
$(2x + 4x) + (3y + 7y)$  Rearrange terms and group like terms using the commutative and associative properties.
$6x + 10y$  Combine like terms.
Simplifying Algebraic Expressions

Simplify: $3(2x + 5) - 7$

$6x + 15 - 7$  
Use the distributive property to remove the parentheses.  

$6x + 9$  
Combine like terms.

Simplify: $3(2a + 4b) + 2(2a + 3b)$

$6a + 12b + 4a + 6b$  
Use the distributive property to remove the parentheses.  

$(6a + 4a) + (12b + 6b)$  
Use the associative and commutative properties to rearrange and group like terms.  

$10a + 18b$  
Combine like terms.
EXAMPLE

Simplify: \(3a - (2a + 4b - 6c) + 2b - 3c\)

SOLUTION

\[3a - (2a + 4b - 6c) + 6b - 3c\]

\[3a - 2a - 4b + 6c + 6b - 3c\]

\[(3a - 2a) + (6b - 4b) + (6c - 3c)\]

\[(3 - 2)a + (6 - 4)b + (6 - 3)c\]

\[1a + 2b + 3c\]

\[a + 2b + 3c\]

Distributive Property
Comm. & Assoc. Prop.
Distributive Property
Subtract
Simplify
§ 1.5

Addition of Real Numbers
The result of adding two or more numbers is called the **sum** of the numbers. You can think of **gains** and **losses** to find sums. For example, what if you had $14 in your pocket, but then lost a ten dollar bill. You would have

\[ +14 + (-10) = 4 \]

Now...another case - what if you lost your wallet with $12 in it and then had to borrow another $5 to get through the day, but then misplaced the $5 also. Your loss of 12 followed by a loss of 5 would be represented by

\[ -12 + (-5) = -17 \]

Now suppose you lost a twenty dollar bill, but then worked and made $16. This would be represented by

\[ (-20) + (+16) = -4 \]
Using the Number Line to Find a Sum

- Let $a$ and $b$ represent real numbers. To find $a + b$ using the number line,

  - Start at $a$.
  
  - If $b$ is positive, move $b$ units to the right.
  - If $b$ is negative, move $b$ units to the left.
  - If $b$ is 0, stay at $a$.

  - The number where we finish on the number line represents the sum of $a$ and $b$. 
Add 7 + (-9) using the number line.

7 + (-9) = -2
Identity and Inverse Properties of Addition

Let $a$ be a real number, a variable or an algebraic expression.

<table>
<thead>
<tr>
<th>Property</th>
<th>Meaning</th>
<th>Examples</th>
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<tbody>
<tr>
<td>Identity Property of Addition</td>
<td>Zero can be deleted from a sum.</td>
<td>$7 + 0 = 7$</td>
</tr>
<tr>
<td></td>
<td>$a + 0 = a$</td>
<td>$0 + 2x = 2x$</td>
</tr>
<tr>
<td></td>
<td>$0 + a = a$</td>
<td>$-3y + 0 = -3y$</td>
</tr>
<tr>
<td>Inverse Property of Addition</td>
<td>The sum of a real number and its additive inverse gives 0, the additive identity.</td>
<td>$5 + (-5) = 0$</td>
</tr>
<tr>
<td></td>
<td>$a + (-a) = 0$</td>
<td>$6x + (-6x) = 0$</td>
</tr>
<tr>
<td></td>
<td>$(-a) + a = 0$</td>
<td>$(-2y) + 2y = 0$</td>
</tr>
</tbody>
</table>
Adding Two Numbers having the Same Sign

1. Add the absolute values.
2. Use the common sign as the sign of the sum.

Example:

-3 + -7 = -10

Add the absolute values: 3 + 7 = 10
Use the common sign.
Adding Two Numbers with Different Signs

1. Subtract the smaller absolute value from the larger absolute value.
2. Use the sign of the number with the greater absolute value.

Example:

\(-13 + 7 = -6\)

- Subtract absolute values: 
  \[13 - 7 = 6\]
- Use the sign of the number with the greater absolute value.
Example: Simplifying Algebraic Expressions with like terms.

Simplify: 

\[-12x + 5x\]
\[(-12 + 5)x\]
\[-7x\]
Adding Real Numbers

EXAMPLE

Add: -12 + (-5)

Answer: -17

We are adding numbers having like signs. So we just add the absolute values and take the common sign as the sign of the sum.

EXAMPLE

Add: -10 + 14

Answer: +4

We are adding numbers having unlike signs. We just take the difference of the absolute values (difference is 4) and then take the sign of the number that has the largest absolute value (that’s the 14 and it is positive).
Rules for Addition of Real Numbers

To add two real numbers with **like signs**, add their absolute values. Use the common sign as the sign of the sum.

To add two real numbers with **different signs**, subtract the smaller absolute value from the greater absolute value. Use the sign of the number with the greater absolute value as the sign of the sum.
§ 1.6

Subtraction of Real Numbers
Definition of Subtraction

• For all real numbers $a$ and $b$,

$$a - b = a + (-b).$$

In words: To subtract $b$ from $a$, add the additive inverse of $b$ to $a$. The result of the subtraction is called the difference.
Subtracting Real Numbers

1. Change the subtraction operation to addition.
2. Change the sign of the number being subtracted.
3. Add, using either the rule for adding numbers having the same sign or adding numbers having different signs.
Examples:

Subtract: $12 - 15$

$12 - 15 = 12 + (-15) = -3$

- Change the sign to addition.
- Replace 15 with -15.

Subtract: $-3 - (-9)$

$-3 - (-9) = -3 + (9) = 6$

- Change the sign to addition.
- Replace the -9 with 9.
Simplifying a Series of Additions and Subtractions

1. Change all subtractions to additions of additive inverses.
2. Group and then add all the positive numbers.
3. Group and then add all the negative numbers.
4. Combine the results of steps 2 and 3.
Example

Series of additions and subtractions:

\[ 7 + 2 - 12 - (-3) \]

\[ = 7 + 2 + (-12) + 3 \] \hspace{1cm} \text{Write as addition of additive inverses.}

\[ = (7 + 2 + 3) + (-12) \] \hspace{1cm} \text{Group the positive numbers and group the negative numbers.}

\[ = 12 + (-12) \] \hspace{1cm} \text{Add the positive numbers.}

\[ = 0 \] \hspace{1cm} \text{Add the results.}
Example

Simplify: \(-5x + 6y - 2x - 7y\)
\[= -5x + 6y + (-2x) + (-7y)\] Write as an addition.
\[= [-5x + (-2x)] + [6y + (-7y)]\] Rearrange terms.
\[= [-5 + (-2)]x + [6 + (-7)]y\] Apply the distributive property.
\[= -7x + (-1)y\] Add within grouping symbols.
Subtracting Real Numbers

Definition of Subtraction

If $a$ and $b$ are real numbers,

$$a - b = a + (-b)$$

That is, to subtract a number, just add its additive opposite (called its additive inverse).
Subtracting Real Numbers

EXAMPLE

Subtract: -12-(-5)

-12-(-5)
-12+5
-7

Here, change the subtraction to addition and replace -5 with its additive opposite. That is, replace the -(-5) with 5.

EXAMPLE

Subtract: -10 - (+4)

-10 +(-4)
-14

Here, change the subtraction to addition and replace +4 with its additive opposite of -4. Then you use the rule for adding two negative numbers.
EXAMPLE

Simplify: \(3a - (5a + 4b - 6c) + 5b - 3c\)

SOLUTION

\[
3a - (5a + 4b - 6c) + 5b - 3c \\
3a - 5a - 4b + 6c + 5b - 3c \\
(3a - 5a) + (5b - 4b) + (6c - 3c) \\
(3 - 5)a + (5 - 4)b + (6 - 3)c \\
-2a + 1b + 3c
\]

Distributive Property
Comm. & Assoc. Prop.
Distributive Property
Subtract and simplify
Difference

Subtraction is used to solve problems in which the word *difference* appears.

The peak of Mount Everest is 8848 meters above sea level.
The Marianas Trench, on the floor of the Pacific Ocean, is 10,915 meters below sea level.

What is the difference in elevation between the peak of Mount Everest and the Marianas Trench?

*Answer: 19,763 meters  Wow...That is *some* difference!*
§ 1.7

Multiplication and Division of Real Numbers
The Product of Two Real Numbers

• The product of two real numbers with different signs is found by multiplying their absolute values. The product is negative.

• The product of two real numbers with same signs is found by multiplying their absolute values. The product is positive.

• The product of 0 and any real number is 0. Thus for any real number $a$,

$$a \cdot 0 = 0 \quad \text{and} \quad 0 \cdot a = 0.$$
Examples: Multiplying Real Numbers

9 \times (5) = 45 \quad \text{Same sign: positive product}

-2 \times (-3) = 6 \quad \text{Same sign: positive product}

-7 \times (4) = 28 \quad \text{Different signs: negative product}

0 \times (7) = 0 \quad \text{Product of 0 and any real number is 0}
Multiplying More Than Two Numbers

1. Assuming that no factor is zero.
   - The product of an *even* number of *negative* numbers is *positive*.
   - The product of an *odd* number of *negative* numbers is *negative*.
   The multiplication is performed by multiplying the absolute values of the given numbers.

2. If any factor is 0, the product is zero.
Multiplying more than two numbers

Examples

Multiply: \((-3)(-5)(2)(-7)\)
\((-3)(-5)(2)(-7) = -210\)

Multiply absolute values: \(3 \cdot 5 \cdot 2 \cdot 7 = 210\)

Odd number of negative numbers: negative product.

Multiply: \((-2)(-4)(6)(-1)(-3)\)
\((-2)(-4)(6)(-1)(-3) = 144\)

Multiply absolute values: \(2 \cdot 4 \cdot 6 \cdot 1 \cdot 3 = 144\)

Even number of negative numbers: positive product.
Definition of Division

If $a$ and $b$ are real numbers and $b$ is not 0, then the quotient of $a$ and $b$ is defined as

$$a \div b = a \cdot \frac{1}{b}$$
Quotient of Two Real Numbers

- The quotient of two real numbers with different signs is found by dividing their absolute values. The quotient is negative.
- The quotient of two real numbers with the same signs is found by dividing their absolute values. The quotient is positive.
- Division of a nonzero number by zero is undefined.
- Any nonzero number divided into 0 is 0.
Division of Real Numbers

Examples:

Divide: \(-\frac{16}{4}\)

\[-\frac{16}{4} = -4\]

Different signs: Negative Quotient

Divide: \(\frac{-12}{-2}\)

\[-\frac{12}{-2} = 6\]

Same signs: Positive Quotient

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Additional Properties of Multiplication

• Identity Property of Multiplication
  \[ a \cdot 1 = a \]
  \[ 1 \cdot a = a \]

• Inverse Property of Multiplication
  If \( a \) is not 0:
  \[ a \cdot \frac{1}{a} = 1 \]
  \[ \frac{1}{a} \cdot a = 1 \]
More Properties of Multiplication

• Multiplication Property of $-1$
  
  \[-1 \cdot a = -a\]

  \[a(-1) = -a\]

• Double Negative Property
  
  The additive inverse of $a$ is $-a$.

  \[-(-a) = a\]
Negative Signs and Parentheses

- If a negative sign precedes a parentheses, remove the parentheses and change the sign of every term within the parentheses.

Examples:

\[-(8x + 7) = -8x + (-7)\]
\[-(-6x - 5) = 6x + 5\]
\[-(-5x + 3) = 5x - 3\]

Note: By doing this, we are using the Distributive Law and distributing the factor of -1.
Example:

Simplify: $3(2x - 5) - (4x + 7)$

$= 3 \cdot 2x - 3 \cdot 5 - 4x - 7$  
Distributive property.

$= 6x - 15 - 4x - 7$  
Multiply.

$= (6x - 4x) + (-15 + -7)$  
Group like terms.

$= 2x + -22$  
Combine like terms.

$= 2x - 22$  
Express addition of an additive inverse as subtraction.

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## Multiplying Real Numbers

<table>
<thead>
<tr>
<th>Rule</th>
<th>Examples</th>
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</thead>
<tbody>
<tr>
<td>The product of two real numbers with <em>different</em> signs is found by multiplying their absolute values. The product is <em>negative</em>.</td>
<td>((-4)8 = -32)</td>
</tr>
<tr>
<td>The product of two real numbers with the <em>same</em> sign is found by multiplying their absolute values. The product is <em>positive</em>.</td>
<td>((-2)(-11) = -22)</td>
</tr>
<tr>
<td>The product of 0 and <em>any</em> real number is 0</td>
<td>(0(-14) = 0)</td>
</tr>
<tr>
<td>If no number is 0, a product with an <em>odd</em> number of negative factors is found by multiplying absolute values. The product is <em>negative</em>.</td>
<td>((-3)(-10)(-6) = -180)</td>
</tr>
<tr>
<td>If no number is 0, a product with an <em>even</em> number of negative factors is found by multiplying absolute values. The product is <em>positive</em>.</td>
<td>(-4(-8)5 = 160)</td>
</tr>
</tbody>
</table>
## Rules for Dividing Real Numbers

<table>
<thead>
<tr>
<th>The quotient of two numbers with <em>different</em> signs is <strong>negative</strong>.</th>
</tr>
</thead>
<tbody>
<tr>
<td>The quotient of two numbers with the <em>same</em> sign is <strong>positive</strong>.</td>
</tr>
</tbody>
</table>

In either *multiplication* or *division* of signed numbers, it is important to **count the negatives** in the product or quotient:
- Odd number of negatives and the answer is negative.
- Even number of negatives and the answer is positive.
§ 1.8

Exponents and Order of Operations
Definition of Natural Number Exponent

If $b$ is a real number and $n$ is a natural number, $b^n$ is read “the $n$th power of $b$” or “$b$ to the $n$th power”.

$$b^n = b \cdot b \cdot b \cdot \ldots \cdot b$$

$b$ appears as a factor $n$ times.

$b$ is the **base** and $n$ is the **exponent**.
Simplifying algebraic expressions that contain exponents.

Simplify: \( 3x^2 + 7x^2 \)
There are two like terms with same variable factor, namely \( x^2 \).
\( 3x^2 + 7x^2 = 10x^2 \)

Simplify: \( 5x^3 + 4x^2 \).
This cannot be simplified because they are not like terms.
Simplifying algebraic expressions that contain exponents.

\[ 3x^4 + 2x^2 + 5x^4 - x^2 = (3x^4 + 5x^4) + (2x^2 - x^2) = 8x^4 + x^2 \]

In this example, you can see that we combined under addition only “like terms”. The variable parts of terms must be the same for them to be “like terms”. Note that we stopped after we had combined like terms. We can not go farther in simplifying in this example.
Order of Operations

1. Perform all operations within grouping symbols. (Starting with innermost.)
2. Evaluate all exponential expressions.
3. Do all multiplications and divisions in the order they occur, working left to right.
4. Finally, do all additions and subtractions in the order in which they occur, working left to right.
Examples: Order of Operation

Simplify : \(9 + 2 \cdot 3\)

\[
= 9 + 6 \quad \text{Do the multiplication first.} \quad (2 \cdot 3 = 6)
\]

\[
= 15 \quad \text{Add.}
\]

Simplify : \(2^3 - 15 \div 5 - 6\)

\[
= 8 - 15 \div 5 - 6 \quad \text{Evaluate the exponential expression.} \quad (2^3 = 8)
\]

\[
= 8 - 3 - 6 \quad \text{Perform the division.} \quad (15 \div 5 = 3)
\]

\[
= 5 - 6 \quad \text{Add and subtract, left to right.}
\]

\[
= -1
\]
Example:

Simplify: 5[2(3-7)+6]

= 5[2(-4)+6] \quad \text{Remove innermost grouping symbol. } 3-7 = -4

= 5[-8+6] \quad \text{Work inside bracket. Multiply } 2 \cdot -4 = -8

= 5[-2] \quad \text{Add inside bracket. } -8 + 6 = -2

= -10 \quad \text{Multiply: } 5[-2] = -10
Example:

Simplify: \(3[5(x - 2) + 8]\)

\[
= 3[5x - 10 + 8]
\]

Remove the innermost grouping symbol using distribution. \(5(x - 2) = 5x - 10\).

\[
= 3[5x - 2]
\]

Add inside the bracket. \(-10 + 8 = -2\)

\[
= 15x - 6
\]

Use the distributive property to remove the bracket. \(3 \cdot 5x - 3 \cdot 2 = 15x - 6\)
In evaluating expressions, what comes first?

• #1 Start with the parentheses. Parentheses say “Me First!”

• #3 Multiplications and divisions are equal in the order of operations – Perform them next.

• #2 Then evaluate the exponential expressions.

• #4 Additions and subtractions are also equal to each other in order – and they come last.

Remember by “PEMDAS” - parentheses, exponents, multiplication, division, addition, subtraction
Order of Operations - PEMDAS

<table>
<thead>
<tr>
<th>Order of Operations</th>
</tr>
</thead>
<tbody>
<tr>
<td>1) First, perform <em>all</em> operations within grouping symbols</td>
</tr>
<tr>
<td>2) Next, Evaluate <em>all</em> exponential expressions.</td>
</tr>
<tr>
<td>3) Next, do all multiplications and divisions in the order in which they occur working from <em>left to right</em>.</td>
</tr>
<tr>
<td>4) Finally, do all additions and subtractions in the order in which they occur, working from <em>left to right</em>.</td>
</tr>
</tbody>
</table>
Order of Operations - PEMDAS

**EXAMPLE**

Evaluate $R^3 - 2R - R$ for $R = 3$.

**SOLUTION**

\[
R^3 - 2R - R \\
3^3 - 2R - 3 \quad \text{Replace R with 3} \\
3^3 - 2R \quad \text{Evaluate inside parentheses first} \\
27 - 2R \quad \text{Evaluate } 3^3 - \text{first exponent}
\]
CONTINUED

27 - 2(81)   Evaluate $3^4$ – second exponent
27 - 162   Multiply
-135   Subtract
Order of Operations

EXAMPLE

Simplify. \( 6 + \frac{4 \cdot 3^2}{6} - 2 \)

SOLUTION

\[
6 + \frac{4 \cdot 3^2}{6} - 2 \\
6 + \frac{4 \cdot 9}{6} - 2 \\
6 + \frac{36}{6} - 2 \\
6 + 6 - 2 \\
-2
\]

Evaluating exponent

Multiply

Divide

Subtract