Chapter 5

Systems of Linear Equations and Inequalities
§ 5.1

Solving Systems of Linear Equations by Graphing
Systems of Equations

We know that an equation of the form $Ax + By = C$ is a line when graphed.

Two such equations is called a system of linear equations.

A solution of a system of linear equations is an ordered pair that satisfies both equations in the system.

For example, the ordered pair $(2,1)$ satisfies the system

\[
\begin{align*}
3x + 2y &= 8 \\
4x - 3y &= 5
\end{align*}
\]
Since two lines may intersect in exactly one point, may not intersect at all, or may intersect in every point; it follows that a system of linear equations will have *exactly one solution*, will have *no solution*, or will have *infinitely many solutions*.
EXAMPLE

Determine whether (3,2) is a solution of the system

\[-3x + 6y = 10\]
\[5x - 8y = -2\]

SOLUTION

Because 3 is the \(x\)-coordinate and 2 is the \(y\)-coordinate of the point (3,2), we replace \(x\) with 3 and \(y\) with 2.

\[-3x + 6y \not= 10\]
\[-3(3) + 6(2) \not= 10\]
\[-9 + 12 \not= 10\]

\[3 = 10 \text{ false}\]

Since the result is false, (3,2) is NOT a solution for the system. Also, I need not check the other equation since the first one failed.
### Solve Systems of Two Linear Equations in Two Variables, $x$ and $y$, by Graphing

1) Graph the first equation.

2) Graph the second equation on the *same set of axes*.

3) If the lines representing the two graphs intersect at a point, determine the coordinates of this point of intersection. The ordered pair is the solution to the system.

4) Check the solution in *both* equations.

**NOTE:** In order for this method to be useful, you must graph the lines *very accurately.*
EXAMPLE

Solve by graphing:  
\[4x + y = 4\]
\[3x - y = 3\]

SOLUTION

1) Graph the first equation. I first rewrite the equation in slope-intercept form.

\[4x + y = 4\]
\[y = -4x + 4\]
\[m = -4 = -4/1, \ b = 4\]

Now I can graph the equation.
CONTINUED

2) Graph the second equation on the same set of axes.
I first rewrite the equation in slope-intercept form.

\[3x - y = 3\]

\[-y = -3x + 3\]

\[y = 3x - 3\]

\[m = 3 = \frac{3}{1}, \hspace{1em} b = -3\]

Now I can graph the equation.
3) Determine the coordinates of the intersection point. This ordered pair is the system’s solution. Using the graph below, it *appears* that the solution is the point (1,0). We won’t know for sure until after we check this potential solution in the next step.
4) Check the solution in both equations.

\[
\begin{align*}
4x + y &= 4 \\
3x - y &= 3 \\
4 + 0 &= 4 \\
3 - 0 &= 3 \\
4 &= 4 \quad \text{true} \\
3 &= 3 \quad \text{true}
\end{align*}
\]

Because both equations are satisfied, \((1,0)\) is the solution and \(\{(1,0)\}\) is the solution set.
EXAMPLE

Solve the system by graphing.

\[ 3x + 2y = 12 \]
\[ 2x - y = 1 \]

Graph the first line. Find the x and y intercepts of \( 3x + 2y = 12 \).

**x-intercept**

Let \( y = 0 \)

\[ 3x + 2(0) = 12 \]

\[ 3x = 12 \]

\[ x = 4 \]

The x-intercept is 4.

**y-intercept**

Let \( x = 0 \)

\[ 3(0) + 2y = 12 \]

\[ 2y = 12 \]

\[ y = 6 \]

The y-intercept is 6.
Graph the first line.
CONTINUED

Now, graph the second line.

Find the x and y intercepts of $2x - y = 1$

<table>
<thead>
<tr>
<th>x-intercept</th>
<th>y-intercept</th>
</tr>
</thead>
<tbody>
<tr>
<td>Let $y = 0$</td>
<td>Let $x = 0$</td>
</tr>
<tr>
<td>$2x - (0) = 1$</td>
<td>$2(0) - y = 1$</td>
</tr>
<tr>
<td>$2x = 1$</td>
<td>$-y = 1$</td>
</tr>
<tr>
<td>$x = 0.5$</td>
<td>$y = -1$</td>
</tr>
</tbody>
</table>

The x-intercept is 0.5 and the y-intercept is -1.
Graph the second line on same set of axes.
Find coordinates of intersection.

Point of intersection: (2,3)
Check the proposed solution in each equation. It appears graphically that the solution is (2,3), but you can’t be sure until you check the point in each of the original equations.

**Equation 1**

\[3x + 2y = 12\]

\[3(2) + 2(3) = 12\]

\[6 + 6 = 12\]

\[12 = 12\]

True

**Equation 2**

\[2x - y = 1\]

\[2(2) - (3) = 1\]

\[4 - 3 = 1\]

\[1 = 1\]

True

Since the point (2,3) checks in each of the original equations, it is indeed the solution of the given system of equations.
EXAMPLE

Solve the system by graphing.

\[ y = 2x - 3 \]
\[ y = 2x + 7 \]

We note that the lines have the same slope, but are not the same line. We know that they are not the same because although the slopes are the same, the y-intercepts are different. Let’s look at the graph of each of the two lines on the same set of axes.
Consider the **first line**. Since the line, $y = 2x - 3$, is in slope-intercept form, we know the slope and y-intercept.
The y-intercept is -3, so the line passes through $(0, -3)$. The slope is $2/1$. To graph, we start at the y-intercept and move 2 up (the rise) and 1 right (the run).

Now, consider the **second line**. Since the line, $y = 2x + 7$, is in slope-intercept form, we know the slope and y-intercept.
The y-intercept is 7, so the line passes through $(0, 7)$. The slope is $2/1$. To graph, we start at the y-intercept and move 2 up (the rise) and 1 right (the run).
CONTINUED

Graph the lines on the same set of axes.

This system is said to be **inconsistent**. Since the lines are parallel and fail to intersect, the system has no solution. We would have to say that the **solution is the empty set**.
### The Number of Solutions to a System of Two Linear Equations

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<tr>
<th>Number of Solutions</th>
<th>What This Means Graphically</th>
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Solving Systems of Equations

Blitzer, *Introductory Algebra*, 5e – Slide #20  Section 5.1
§ 5.2

Solving Systems of Linear Equations by the Substitution Method
Finding the solution of a system graphically is not always easy to do. Suppose the coordinates of the point of intersection were fractions, or even worse, irrational numbers.

In this section, we consider a method that does not depend on finding a systems solutions visually. This method involves converting the system of two equations in two variables to one equation in one variable by using an appropriate substitution. Thus this method is called “Substitution.”
Steps for Solving Systems Using Substitution

1. Solve either of the equations for one variable in terms of the other.
2. Substitute the expression from step 1 into the other equation.
3. Solve the resulting equation.
4. Back-substitute the obtained value into the equation from step 1.
5. Check the solution in both of the original equations.
Solve the System Using Substitution.

\[ y = 2x - 8 \]
\[ 2x + 3y = 16 \]

**Step 1: Solve either of the equations for one variable in terms of the other.**

Note that the first equation is already solved for \( y \), so we will use that.
Continued

Solve using substitution: \( y = 2x - 8 \)
\( 2x + 3y = 16 \)

Step 2: Substitute the expression from step 1 into the other equation.

\( 2x + 3(2x - 8) = 16 \)

Substitute \( 2x - 8 \) for \( y \)
Step 3: Solve the resulting equation.

\[ 2x + 3(2x - 8) = 16 \]
\[ 2x + 6x - 24 = 16 \]
\[ 8x - 24 = 16 \]
\[ 8x = 40 \]
\[ x = 5 \]
Step 4: Back-substitute the obtained value into the equation from step 1.
Substitute $x = 5$ into the first equation.

$y = 2(5) - 8$
$y = 10 - 8$
$y = 2$

The proposed solution is $x = 5$ and $y = 2$ or the ordered pair $(5, 2)$
Solve using substitution: $y = 2x - 8$

$2x + 3y = 16$

**Step 5: Check the solution in both equations.**

$(5, 2)$

<table>
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<th>Equation 2</th>
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<tr>
<td>$2 \ ? 2(5) - 8$</td>
<td>$2(5) + 3(2) \ ? 16$</td>
</tr>
<tr>
<td>$2 \ ? 10 - 8$</td>
<td>$10 + 6 \ ? 16$</td>
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</table>

$2 = 2$

$16 = 16$

True

True

$(5, 2)$ satisfies both equations. The system’s solution is $(5, 2)$. 

Blitzer, *Introductory Algebra*, 5e – Slide #28 Section 5.2
### Substitution Method

#### Solving Linear Systems by Substitution

1. Solve either of the equations for one variable in terms of the other. *(If one of the equations is already in this form, you can skip this step)*

2. Substitute the expression found in step 1 into the other equation. This will result in an equation in one variable! (Hence, it’s called the “Substitution Method”)

3. Solve the equation containing one variable. (the resultant equation from step 2)

4. Back-Substitute the value found in step 3 into one of the original equations. Simplify and find the value of the remaining variable.

5. Check the proposed solution in both of the system’s given equations.
EXAMPLE

Solve by the substitution method:

\[ 4x + y = 4 \]
\[ 3x - y = 3 \]

SOLUTION

1) Solve either of the equations for one variable in terms of the other. We’ll isolate the variable \( y \) from the first equation.

\[ 4x + y = 4 \]
\[ y = -4x + 4 \]

Solve for \( y \) by subtracting 4x from both sides.
Substitution Method

CONTINUED

2) Substitute the expression from step 1 into the other equation. 

\(-4x + 4\) is the ‘expression from step 1’ and ‘the other equation’ is \(3x - y = 3\). Therefore:

\[
3x - y = 3 \\
3x - (-4x + 4) = 3
\]

Replace \(y\) with \(-4x + 4\)

3) Solve the resulting equation containing one variable.

\[
3x + 4x - 4 = 3 \\
7x - 4 = 3 \\
7x = 7 \\
x = 1
\]

Distribute  
Add like terms  
Add 4 to both sides  
Divide both sides by 7
Substitution Method

4) Back-substitute the obtained value into one of the original equations. We back-substitute 1 for \( x \) into one of the original equations to find \( y \). Let’s use the first equation.

\[
4x + y = 4
\]

\[
4 \cdot 1 + y = 4 \quad \text{Replace } x \text{ with 1}
\]

\[
4 + y = 4 \quad \text{Multiply}
\]

\[
y = 0 \quad \text{Subtract 4 from both sides}
\]

Therefore, the potential solution is (1,0).
Substitution Method

CONTINUED

5) Check. Now we will show that (1,0) is a solution for both of the original equations.

\[ 4x + y = 4 \quad 3x - y = 3 \]

4 \( \square \) 0 \( \neq \) 4 \quad 3 \( \square \) 0 \( \neq \) 3

4 + 0 \( \neq \) 4 \quad 3 - 0 \( \neq \) 3

4 = 4 \quad 3 = 3 \quad \text{true}

Because both equations are satisfied, (1,0) is the solution and \{(1,0)\} is the solution set.
The Number of Solutions to a System of Two Linear Equations

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EXAMPLE

At a price of $p$ dollars per ticket, the number of tickets to a rock concert that can be sold is given by the demand model $N = -25p + 7800$. At a price of $p$ dollars per ticket, the number of tickets that the concert’s promoters are willing to make available is given by the supply model $N = 5p + 6000$.

(a) How many tickets can be sold and supplied for $50 per ticket?

(b) Find the ticket price at which supply and demand are equal. At this price, how many tickets will be supplied and sold?
Solving Systems of Equations

CONTINUED

SOLUTION

(a) How many tickets can be sold and supplied for $50 per ticket?

The number of tickets that can be sold for $50 per ticket is found using the demand model:

\[ N = -25(50) + 7800 = -1250 + 7800 = 6550 \text{ tickets sold.} \]

The number of tickets that can be supplied for $50 per ticket is found using the supply model:

\[ N = 5(50) + 6000 = 250 + 6000 = 6250 \text{ tickets supplied.} \]
(b) Find the ticket price at which supply and demand are equal. At this price, how many tickets will be supplied and sold?

1) Solve either of the equations for one variable in terms of the other. The system of equations already has $N$ isolated in both equations.

\[ N = -25p + 7800 \]
\[ N = 5p + 6000 \]
2) Substitute the expression from step 1 into the other equation. I’ll replace the $N$ in the second equation with $-25p + 7800$.

\[-25p + 7800 = 5p + 6000\]

3) Solve the resulting equation containing one variable.

\[-25p + 7800 = 5p + 6000\]

\[7800 = 30p + 6000\]  \hspace{1cm} \text{Add 25$p$ to both sides}

\[1800 = 30p\]  \hspace{1cm} \text{Subtract 6000 from both sides}

\[60 = p\]  \hspace{1cm} \text{Divide both sides by 30}
Substitution Method

CONTINUED

4) Back-substitute the obtained value into one of the original equations. We back-substitute 60 for $p$ into one of the original equations to find $N$. Let’s use the first equation.

\[ N = -25p + 7800 \]
\[ N = -25(60) + 7800 \]  Replace $p$ with 60
\[ N = -1500 + 7800 \]  Multiply
\[ N = 6300 \]  Add
Substitution Method

CONTINUED

5) Check. Now we will show that (60,6300) is a solution for both of the original equations.

\[ N = -25p + 7800 \quad \quad \quad N = 5p + 6000 \]

\[
\begin{align*}
6300 & \overset{?}{=} -25(60) + 7800 \\
6300 & \overset{?}{=} -1500 + 7800 \\
6300 & = 6300 \quad \text{true} \\
6300 & \overset{?}{=} 5(60) + 6000 \\
6300 & \overset{?}{=} 300 + 6000 \\
6300 & = 6300 \quad \text{true}
\end{align*}
\]

Therefore, the solution is (60,6300). Therefore, supply and demand will be equal when the ticket price is $60 and 6300 tickets are sold.
§ 5.3

Solving Systems of Linear Equations and Inequalities
The third method we will consider for solving a system of equations is the Addition Method.

When we use the **addition method**, we will again try to eliminate one variable, but we will do that by adding the equations.

**Frequently**, the addition method is the **easiest** method for solving a system of linear equations.

We saved the best for last.
Steps in the Addition Method

1. If necessary, rewrite both equations in the form $Ax + By = C$.

2. If necessary, multiply either or both equations by appropriate nonzero numbers so that the sum of either the $x$-coefficients or the sum of the $y$-coefficients is 0.

3. Add the equations from step 2.

4. Solve the equation from step 3.

5. Back substitute the value from step 4 into either of the original equations and solve for the other variable.

6. Check the solution.
Addition (Elimination) Method

EXAMPLE

Solve by the addition method:

\[ 4x = -2y + 4 \]
\[-y = -3x + 3 \]

SOLUTION

1) Rewrite both equations in the form \( Ax + By = C \). We first arrange the system so that variable terms appear on the left and constants appear on the right. We obtain

\[ 4x + 2y = 4 \quad \text{Add 2y to both sides} \]
\[ 3x - y = 3 \quad \text{Add 3x to both sides} \]
Addition (Elimination) Method

2) If necessary, multiply either equation or both equations by appropriate numbers so that the sum of the \(x\)-coefficients is 0. We can eliminate the \(y\)’s by multiplying the second equation by 2. Or we can eliminate the \(x\)’s by multiplying the first equation by -3 and the second equation by 4. Let’s use the first method.

\[
\begin{align*}
4x + 2y &= 4 & \text{(No Change)} \\
3x - y &= 3 & \text{(Multiply by 2)}
\end{align*}
\]

\[
\begin{align*}
4x + 2y &= 4 \\
6x - 2y &= 6
\end{align*}
\]
CONTINUED

3) Add the equations.

\[ 4x + 2y = 4 \]
\[ 6x - 2y = 6 \]
\[
\begin{array}{c}
\text{Add:} \\
10x + 0y = 10 \\
10x = 10
\end{array}
\]

4) Solve the equation in one variable. Now I solve the equation \( 10x = 10 \).

\[
10x = 10
\]

\[
x = 1
\]
5) Back-substitute and find the value of the other variable. Now we will use one of the original equations and replace $x$ with 1 to determine $y$. I’ll use the second equation.

\[-y = -3x + 3\]
\[-y = -3(1) + 3\] Replace $x$ with 1
\[-y = -3 + 3\] Multiply
\[-y = 0\] Add
\[y = 0\] Multiply by -1

Therefore, the potential solution is $(1,0)$. 
6) **Check.** I now check the potential solution (1,0) in both original equations.

\[
\begin{align*}
4x &= -2y + 4 \\
-2(0) + 4 &= 4
\end{align*}
\]

\[
\begin{align*}
-3x + 3 &= -y \\
-3(1) + 3 &= 0
\end{align*}
\]

\[
\begin{align*}
4 &= 0 + 4 \\
0 &= 0 + 3
\end{align*}
\]

\[
\begin{align*}
4 &= 4 \quad \text{true} \\
0 &= 0 \quad \text{true}
\end{align*}
\]

Because both equations are satisfied, (1,0) is the solution and \{(1,0)\} is the solution set.
Addition Method

EXAMPLE

Solve the system using the addition method.

\[ 3x - y = 27 \]
\[ 4x + y = 8 \]

Step 1: If necessary, rewrite both equations in the form \( Ax + By = C \).

Not necessary, since the equations are already in \( Ax + By = C \) form.
CONTINUED

Solve using addition: $3x - y = 27$
$4x + y = 8$

Step 2: If necessary, multiply either or both equations by appropriate nonzero numbers so that the sum of either the $x$-coefficients or the sum of the $y$-coefficients is 0.

Not necessary here, since the sum of the $y$-coefficients is 0
CONTINUED

Solve using addition: $3x - y = 27$

Step 3: Add the equations from step 2.

$3x - y = 27$

$4x + y = 8$

$7x = 35$
CONTINUED

Solve using addition: \(3x - y = 27\)
\[4x + y = 8\]

**Step 4: Solve the equation from step 3.**

\[7x = 35\]
\[x = 5\]
Solve using addition: \(3x - y = 27\)
\(4x + y = 8\)

**Step 5: Back-substitute and solve for the other variable.**

Substitute \(x = 5\) into either of the original equations and solve for \(y\).

\[3(5) - y = 27\]
\[15 - y = 27\]
\[- y = 12\]
\[y = -12\]

The solution is \((5, -12)\)
Solve using addition:

\[ 3x - y = 27 \]
\[ 4x + y = 8 \]

**Step 6: Check the solution in both equations.**

**Equation 1:**
\[ 3(5) - (-12) \overset{?}{=} 27 \]
\[ 15 + 12 \overset{?}{=} 27 \]
\[ 27 = 27 \]
\[ \text{True.} \]

**Equation 2:**
\[ 4(5) + (-12) \overset{?}{=} 8 \]
\[ 20 + (-12) \overset{?}{=} 8 \]
\[ 8 = 8 \]
\[ \text{True.} \]

The solution is \((5, -12)\)
EXAMPLE

Solve the system using the addition method.

\[ 2x + 3y = 6 \]
\[ 4y = 13 - x \]

Step 1: If necessary, rewrite both equations in the form \( Ax + By = C. \)

\[ 2x + 3y = 6 \]
\[ x + 4y = 13 \]
CONTINUED

Solve using addition: $2x + 3y = 6$
$4y = 13 - x$

Step 2: If necessary, multiply either or both equations by appropriate nonzero numbers so that the sum of either the $x$-coefficients or the sum of the $y$-coefficients is 0.

Multiply second equation by (-2) so the sum of the $x$-coefficients is 0. (Both sides!)

$$-2(x + 4y) = -2 (13) \text{ leads to the equation } -2x - 8y = -26$$
Solve using addition: \(2x + 3y = 6\)

\(4y = 13 - x\)

**Step 3: Add the equations from step 2.**

\[\begin{align*}
2x + 3y &= 6 \\
-2x - 8y &= -26
\end{align*}\]

\[-5y = -20\]
Solve using addition: $2x + 3y = 6$

$4y = 13 - x$

**Step 4: Solve the equation from step 3.**

$-5y = -20$

$y = 4$
Solve using addition: \( 2x + 3y = 6 \)
\( 4y = 13 - x \)

Step 5: Back-substitute and solve for the other variable.
Substitute \( y = 4 \) into one of the original equations and solve for \( x \).
\[
2x + 3(4) = 6 \\
2x + 12 = 6 \\
2x = -6 \\
x = -3
\]
The solution is \((-3, 4)\).
Solve using addition: \(2x + 3y = 6\)

\(4y = 13 - x\)

**Step 6: Check the solution in both equations.**

**Equation 1:**
\[
2(-3) + 3(4) \leq 6
\]
\[
-6 + 12 \leq 6
\]
\[
6 = 6
\]

True.

**Equation 2:**
\[
4(4) \leq 13 - (-3)
\]
\[
16 \leq 13 + 3
\]
\[
16 = 16
\]

True.

The solution is \((-3, 4)\).
Solving Linear Systems by Addition

1) If necessary, rewrite both equations in the form $Ax + By = C$.

2) If necessary, multiply either equation or both equations by appropriate nonzero numbers so that the sum of the $x$-coefficients or the sum of the $y$-coefficients is 0.

3) Add the equations in step 2. The sum should be an equation in one variable.

4) Solve the equation in one variable (the result of step 3).

5) Back-substitute the value obtained in step 4 into either of the given equations and solve for the other variable.

6) Check the solution in both of the original equations.

*NOTE: As you now know, there is more than one method to solve a system of equations. The reason for learning more than one method is because sometimes one method will be preferable or easier to use over another method.*
## Comparing Solution Methods

<table>
<thead>
<tr>
<th>Method</th>
<th>Advantages</th>
<th>Disadvantages</th>
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<tbody>
<tr>
<td>Graphing</td>
<td>You can see the solutions.</td>
<td>If the solutions do not involve integers or are too large to be seen on the graph, it’s impossible to tell exactly what the solutions are.</td>
</tr>
<tr>
<td>Substitution</td>
<td>Gives exact solutions. Easy to use <em>if a variable is on one side by itself.</em></td>
<td>Solutions cannot be seen. Introduces extensive work with fractions when no variable has a coefficient of 1 or -1.</td>
</tr>
<tr>
<td>Addition</td>
<td>Gives exact solutions. Easy to use <em>if a variable has a coefficient of 1 or -1.</em></td>
<td>Solutions cannot be seen.</td>
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NOTE: It is *extremely* helpful to understand these relationships as well as any other relationship between an equation and its graph.

NOTE: To determine that a system has exactly one solution, solve the system using one of the methods. A single solution will occur as in the previous examples.

NOTE: To determine that a system has no solution, solve the system using one of the methods. Eventually, you’ll get an *obviously false statement*, like $3 = 4$.

NOTE: To determine that a system has infinitely many solutions, solve the system using one of the methods. Eventually, you’ll get an *obviously true statement*, like $-2 = -2$. 
§ 5.4

Problem Solving and Business Applications Using Systems of Equations
The Strategy

1. Use variables to represent unknown quantities.
2. Write a system of equations describing the problems conditions.
3. Solve the system and answer the problem’s questions.
4. Check the proposed answers in the original wording of the problem.
EXAMPLE

Burgers and Fries

Two burgers and one order of fries contain 34 grams of fat.
Two orders of fries and one burger contain 41 grams of fat.
Find the number of grams of fat in each item.

Step 1: Use variables to represent unknown quantities.

Let $B =$ number of fat grams in one burger
$F =$ number of fat grams in one order of fries
Step 2: Write a system of equations describing the problem’s conditions.

2B + 1F = 34
B + 2F = 41

34 grams of fat.
41 grams of fat

Note that these sentences are all about fat… Fat + more Fat = Total Fat
Step 3: Solve the system and answer the problem’s question.

\[ 2B + 1F = 34 \]
\[ B + 2F = 41 \]

We can use either substitution or the addition property to solve the system.
Burgers and Fries

\[2B + F = 34\]
\[B + 2F = 41\]

Let’s use Substitution.

Solve the first equation for \(F\).

\[F = 34 - 2B\]

Substitute into equation 2.

\[B + 2(34 - 2B) = 41\]
Burgers and Fries

\[ B + 2(34 - 2B) = 41 \]

Solve for \( B \):

\[ B + 68 - 4B = 41 \]

\[-3B + 68 = 41 \]

\[-3B = -27 \]

\[ B = 9 \]
If $B = 9$

$F = 34 - 2B = 34 - 9(2)$

$F = 34 - 18$

$F = 16$

There are 9 grams of fat in a burger and 16 grams of fat in an order of fries.
Our proposed solution: Burgers 9 grams of fat, Fries 16 grams of fat

**Step 4:** Check the proposed solution in the original wording of the problem.

Two burgers and one order of fries contain 34 grams of fat. Two orders of fries and one burger contain 41 grams of fat. Find the number of grams of fat in each item.

Let’s see...Two burgers would have 18 grams plus one fry with 16 grams would total 34 grams. And two fries would have 32 grams plus one burger with 9 grams would total 41 grams. So we got it right... But ug, all that fat – better start eating at home now! Did you know before you set up the system that the fries have more fat? How did you know that from the information given?
Think again about our strategy here…

### A Strategy for Solving Word Problems

1) Use variables to represent unknown quantities.

2) Write a system of equations describing the problem’s conditions (translate from English to Math Language).

3) Solve the system and answer the problem’s question (make sure you answer the question being asked, \textit{not some other question!}).

4) Check the proposed solution \textit{in the original wording of the problem}.
EXAMPLE

You invested $7000 in two accounts paying 6% and 8% annual interest, respectively. If the total interest earned for the year was $520, how much was invested at each rate?

SOLUTION

1) Use variables to represent unknown quantities.

Let $x = \text{the amount invested at 6\%}$.  
Let $y = \text{the amount invested at 8\%}$. 
2) Write a system of equations describing the problem’s conditions.

\[ x + y = 7000 \]

This tells us the sum of the money invested is $7000. Also, we’re concerned with how much money was invested into each of the accounts. We can use a table to organize the information in the problem and obtain a second equation.

<table>
<thead>
<tr>
<th>Principal (Amount Invested)</th>
<th>Interest Rate</th>
<th>Interest Earned</th>
</tr>
</thead>
<tbody>
<tr>
<td>6% Investment</td>
<td>( x )</td>
<td>0.06</td>
</tr>
<tr>
<td>8% Investment</td>
<td>( y )</td>
<td>0.08</td>
</tr>
</tbody>
</table>
The amount of interest from each account is:

\[0.06x \text{ and } 0.08y.\]

And since we are concerned with both amounts of interest adding up to $520, the following equation results:

\[0.06x + 0.08y = 520.\]

3) Solve the system and answer the problem’s question.

The system

\[x + y = 7000\]

\[0.06x + 0.08y = 520\]
Systems of Equations in Application

CONTINUED

can be solved by substitution or addition. Substitution will work well since both variables in the first equation have coefficients of 1. So, we’ll use substitution. First we’ll isolate $x$ in the first equation.

\[ x + y = 7000 \]
\[ x = 7000 - y \quad \text{Subtract } y \text{ from both sides} \]

Now substitute $7000 - y$ for $x$ in the second equation.
Systems of Equations in Application

CONTINUED

\[ 0.06x + 0.08y = 520 \]
\[ 0.06(7000 - y) + 0.08y = 520 \]
\[ 420 - 0.06y + 0.08y = 520 \]
\[ 420 + 0.02y = 520 \]
\[ 0.02y = 100 \]
\[ y = 5000 \]

Replace \( x \) with \( 7000 - y \)

Distribute

Add like terms

Subtract 100 from both sides

Divide both sides by 0.02

Now substitute 5000 for \( y \) in either of the original equations. We’ll use the first.
CONTINUED

\[ x + y = 7000 \]
\[ x + (5000) = 7000 \quad \text{Replace } y \text{ with } 7000 \]
\[ x = 2000 \quad \text{Subtract } 5000 \text{ from both sides} \]

Therefore, $2000 will be invested into the 6% account and $5000 will be invested in the 8% account. This is the potential solution.
4) Check the proposed answer in the original wording of the problem.

Do both investments add up to $7000?

\[ x + y = 2000 + 5000 = 7000 \quad \text{true} \]

Do the two investments yield a combined $520 in interest? The 6\% investment yields, in interest, \( 0.06x = 0.06(2000) = 120 \). The 8\% investment yields, in interest, \( 0.08y = 0.08(5000) = 400 \). And since $120 + $400 = $520, the solution has been verified.
EXAMPLE

A jeweler needs to mix an alloy with a 16% gold content and an alloy with a 28% gold content to obtain 32 ounces of a new alloy with a 25% gold content. How many ounces of each of the original alloys must be used?

SOLUTION

1) Use variables to represent unknown quantities.

Let $x =$ the number of ounces of the 16% alloy to be used in the mixture.

Let $y =$ the number of ounces of the 28% alloy to be used in the mixture.
2) Write a system of equations describing the problem’s conditions. We need 32 ounces of an alloy containing 25% gold content. We form a table that shows the amount of gold content in each of the three alloys.

<table>
<thead>
<tr>
<th></th>
<th>Number of Ounces</th>
<th>Percent of Gold Content</th>
<th>Amount of Gold Content</th>
</tr>
</thead>
<tbody>
<tr>
<td>16% Alloy</td>
<td>$x$</td>
<td>16% = 0.16</td>
<td>0.16$x$</td>
</tr>
<tr>
<td>28% Alloy</td>
<td>$y$</td>
<td>28% = 0.28</td>
<td>0.28$y$</td>
</tr>
<tr>
<td>25% Alloy</td>
<td>32</td>
<td>25% = 0.25</td>
<td>32(0.25) = 8</td>
</tr>
</tbody>
</table>
Systems of Equations in Application

CONTINUED

Since adding the two amounts of alloy \((x\) and \(y\)) will yield 32 ounces of 25% alloy,

\[ x + y = 32. \]

The 32-ounce mixture must have a 25% gold content. The amount of gold content must be 25% of 32 ounces, or \((0.25)(32) = 8\) ounces.

\[ 0.16x + 0.28y = 8 \]

Therefore, the system of equations is

\[ x + y = 32 \]
\[ 0.16x + 0.28y = 8 \]
CONTINUED

NOTE: The first equation summarizes how many total ounces of each of the three alloys. The second equation summarizes how much gold content in each of the three solutions. *Each equation summarizes exactly one type of quantity.*

3) Solve the system and answer the problem’s question.
   The system can be solved by substitution or addition. Substitution will work well since both variables in the first equation have coefficients of 1. So, we’ll use substitution. First we’ll isolate \( y \) in the first equation.
CONTINUED

\[ x + y = 32. \]
\[ y = 32 - x \]  \hspace{1cm} \text{Subtract } x \text{ from both sides}

Now substitute \( 32 - x \) for \( y \) in the second equation.

\[ 0.16x + 0.28y = 8 \]
\[ 0.16x + 0.28(32 - x) = 8 \]  \hspace{1cm} \text{Replace } y \text{ with } 32 - x
\[ 0.16x + 8.96 - 0.28x = 8 \]  \hspace{1cm} \text{Distribute}
\[ 8.96 - 0.12x = 8 \]  \hspace{1cm} \text{Add like terms}
\[ -0.12x = -0.96 \]  \hspace{1cm} \text{Subtract } 8.96 \text{ from both sides}
\[ x = 8 \]  \hspace{1cm} \text{Divide both sides by } -0.12
Systems of Equations in Application

CONTINUED

Now substitute 8 for $x$ in either of the original equations. We’ll use the first.

\[ x + y = 32 \]
\[ 8 + y = 32 \quad \text{Replace } x \text{ with } 8 \]
\[ y = 24 \quad \text{Subtract 8 from both sides} \]

Therefore, the potential solution is (8,24). That is, 8 ounces of the 160% alloy and 24 ounces of the 28% alloy.
CONTINUED

4) Check the proposed answer in the original wording of the problem.

Do the amounts of the two alloys add up to the desired 32 ounces?

\[ x + y = 8 + 24 = 32 \quad \text{true} \]

Also, the problem states that we need 32 ounces of a 25% gold content alloy. The amount of gold content in this mixture is \((32)(0.25) = 8\) ounces of gold content. The amount of gold content in 8 ounces of a 16% solution is \((8)(0.16) = 1.28\) ounces. The amount of gold content in 24 ounces of a 28% alloy is \((24)(0.28) = 6.72\) ounces. So, \(1.28 + 6.72 = 8\) ounces of gold content, exactly as it should be.
EXAMPLE

A motorboat traveled 36 miles downstream, with the current, in 1.5 hours. The return trip upstream, against the current, covered the same distance, but took 2 hours. Find the boat’s rate in still water and the rate of the current.

SOLUTION

1) Use variables to represent unknown quantities.

Let $x =$ the rate (speed) of the motorboat.
Let $y =$ the rate (speed) of the stream.
2) Write a system of equations describing the problem’s conditions.

<table>
<thead>
<tr>
<th></th>
<th>Rate</th>
<th>Time</th>
<th>Distance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Downstream</td>
<td>$x + y$</td>
<td>1.5</td>
<td>36</td>
</tr>
<tr>
<td>Upstream</td>
<td>$x - y$</td>
<td>2</td>
<td>36</td>
</tr>
</tbody>
</table>
Systems of Equations in Application

CONTINUED

Therefore, the system of equations is:

\[(x + y)1.5 = 36\]
\[(x - y)2 = 36\]

3) Solve the system and answer the problem’s question.

Upon distributing, the system simplifies to

\[(x + y)1.5 = 36\] \hspace{1cm} \[(x - y)2 = 36\]
\[1.5x + 1.5y = 36\] \hspace{1cm} \[2x - 2y = 36\]

I will now solve the second equation for \(x\).

\[2x - 2y = 36\] \hspace{1cm} \[2x = 36 + 2y\] \hspace{1cm} \[x = 18 + y\]

Second equation \hspace{1cm} Add 2y \hspace{1cm} Divide by 2
Systems of Equations in Application

CONTINUED

In the first equation, replace $x$ with $18 + y$.

\[ 1.5x + 1.5y = 36 \]
\[ 1.5(18 + y) + 1.5y = 36 \]
\[ 27 + 1.5y + 1.5y = 36 \]
\[ 27 + 3y = 36 \]
\[ 3y = 9 \]
\[ y = 3 \]

First equation
Replace $x$ with $18 + y$
Distribute
Combine like terms
Subtract 27 from both sides
Divide both sides by 3

Now we can use either equation to find $x$. Let’s use the second equation.
CONTINUED

\[ 2x - 2y = 36 \quad \text{Second equation} \]
\[ 2x - 2(3) = 36 \quad \text{Replace } y \text{ with 3} \]
\[ 2x - 6 = 36 \quad \text{Multiply} \]
\[ 2x = 42 \quad \text{Add 6 to both sides} \]
\[ x = 21 \quad \text{Divide both sides by 2} \]

Therefore, the potential solution is (21,3).
CONTINUED

4) Check the proposed answer in the original wording of the problem.

I now verify that my solutions satisfy the original equations:

\[
\begin{align*}
1.5x + 1.5y &= 36 \\
2x - 2y &= 36 \\
1.5(21) + 1.5(3) &= 36 \\
2(21) - 2(3) &= 36 \\
31.5 + 4.5 &= 36 \\
42 - 6 &= 36 \\
36 &= 36 \quad \text{true} \\
36 &= 36 \quad \text{true}
\end{align*}
\]

So, the motorboat travels 21 miles per hour in still water and the speed of the current is 3 miles per hour.
Revenue, Cost and Profit

Financial Functions

<table>
<thead>
<tr>
<th>Function</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>Revenue Function</td>
<td>$R(x) = \text{(price per unit sold) } x$</td>
</tr>
<tr>
<td>Cost Function</td>
<td>$C(x) = \text{fixed cost} + \text{(cost per unit produced) } x$</td>
</tr>
<tr>
<td>Profit Function</td>
<td>$P(x) = R(x) - C(x)$</td>
</tr>
</tbody>
</table>

The point of intersection of the revenue and cost functions is called the break-even point. The x-coordinate of the point reveals the number of units that a company must produce and sell so that the money coming in is equal to the money going out. That is, the break-even point is where revenue $=$ cost.
EXAMPLE

You invested $30,000 and started a business writing greeting cards. Supplies cost 2 cents per card and you are selling each card for 50 cents. (In solving this exercise, let x represent the number of cards produced and sold.)

(a) Write the cost function, \( C \).

(b) Write the revenue function, \( R \).

(c) Determine the break-even point. Describe what this means.

SOLUTION

(a) Write the cost function, \( C \).

\[
C(x) = 30,000 + 0.02x
\]
(b) Write the revenue function, \( R \).

\[ R(x) = 0.50x \]

(c) Determine the break-even point. Describe what this means.

The break-even point occurs when revenue and cost are equal. That is,
Therefore, the break-even point will occur when 62,500 cards are sold. When that happens, revenue and cost will be equal. For any number of units over 62,500 sold, the company will make a profit.

\[ R(x) = C(x) \]
\[ 0.5x = 30,000 + 0.02x \]
\[ 0.48x = 30,000 \]
\[ x = 62,500 \]

Subtract 0.02x from both sides
Divide both sides by 0.48
§ 5.5

Systems of Linear Inequalities
The **solution set of a system of linear inequalities in two variables** is the set of all ordered pairs that satisfy each inequality in the system.

To graph a system of inequalities in two variables, *begin* by graphing each individual inequality on the same graph. Then find the region, if there is one, that is common to every graph in the system.

This **region of intersection** or region of overlap gives the system’s solution set.
Graph each inequality in the same coordinate system.

Find the region, if there is one, that is common to every graph in the system. This is the region of overlap.
First recall how to graph a single linear inequality in two variables.

### Graphing a Linear Inequality in Two Variables

1) Replace the inequality symbol with an equal sign and graph the corresponding linear equation. Draw a solid line if the original inequality contains a ≤ or ≥ symbol. Draw a dashed line if the original inequality contains a < or > symbol.

2) Choose a test point (in one of the half-planes) that is not on the line. Substitute the coordinates of the test point into the inequality.

3) If a true statement results, shade the half-plane containing this test point. If a false statement results, shade the half-plane not containing this test point.
EXAMPLE

Graph the solution of the system:  
\[ 5x + 2y \leq 10 \]
\[ 2x - y \geq 4 \]

First graph \(5x + 2y \leq 10\).

x-intercept (2, 0), y-intercept (0, 5)

Check point (0, 0). True.
Graph the solution: \(5x + 2y \leq 10\)
\(2x - y \geq 4\)

Then graph the second half plane: \(2x - y \leq 4\).

x-intercept (2, 0), y-intercept (0, -4)

Check point (0, 0). False.
Graphing a System of Linear Inequalities

CONTINUED

Graph the solution:  

\[ 5x + 2y \leq 10 \]
\[ 2x - y \geq 4 \]

Solution is the overlap of

\[ 5x + 2y \leq 10 \] and \[ 2x - y \geq 4 \]
Graph the solution: $5x + 2y \leq 10$

$2x - y \geq 4$

Solution is the overlap of

$5x + 2y \leq 10$ and $2x - y \geq 4$. 

Overlap
Graph the solution: $5x + 2y \leq 10$

$2x - y \geq 4$

Note: The solution contains all of the green, including borders.
EXAMPLE

Graph the solution set of the system.

\[ x + y > 3 \]
\[ x + y > -2 \]

SOLUTION

Replacing each inequality symbol with an equal sign indicates that we need to graph \( x + y = 3 \) and \( x + y = -2 \). We can use intercepts to graph these lines.

\[ x + y = 3 \]
\[ x + 0 = 3 \]
\[ x = 3 \]

\[ x + y = -2 \]
\[ x + 0 = -2 \]
\[ x = -2 \]

Replace \( y \) with 0

Simplify

The two \( x \)-intercepts are \((3,0)\) and \((-2,0)\).
CONTINUED

\[ x + y = 3 \quad \quad x + y = -2 \]
\[ 0 + y = 3 \quad \quad 0 + y = -2 \quad \text{Replace } x \text{ with } 0 \]
\[ y = 3 \quad \quad y = -2 \quad \text{Simplify} \]

The two \( y \)-intercepts are \((0,3)\) and \((0,-2)\).

Graph \( x + y > 3 \). The blue line, \( x + y = 3 \), is dashed: Equality is not included in \( x + y > 3 \). Because \((0,0)\) makes the inequality false, we shade the half-plane not containing \((0,0)\) in green.
Add the graph of \( x + y > -2 \). The red line, \( x + y = -2 \), is dashed: Equality is not included in \( x + y > -2 \). Because \((0,0)\) makes the inequality true, we shade the half-plane containing \((0,0)\) using yellow horizontal shading.
The solution set of the system is graphed as the intersection (the overlap) of the two half-planes. This is the region in which the green vertical shading and the yellow horizontal shading overlap. Notice that the blue line is not part of the solution since it is not contained within both solution sets.
EXAMPLE

(a) An elevator can hold no more than 2000 pounds. If children average 80 pounds and adults average 160 pounds, write a system of inequalities that models when the elevator holding $x$ children and $y$ adults is overloaded.

(b) Graph the solution set of the system of inequalities in part (a).

SOLUTION

(a) Since the number of children on the elevator will always be zero or more, we get the inequality: $x \geq 0$.

Since the number of adults on the elevator will always be zero or more, we get the inequality: $y \geq 0$. 
Graphing a System of Linear Inequalities

CONTINUED

The number of pounds that children contribute to the weight in the elevator is \((\text{the average weight of a child}) \times (\text{the number of children}) = 80x\). The number of pounds that adults contribute to the weight in the elevator is \((\text{the average weight of an adult}) \times (\text{the number of adults}) = 160y\). Now, the sum of these two quantities cannot exceed 2000 pounds. Therefore, the inequality \(80x + 160y > 2000\) would tell us when the elevator is overloaded.

So, the system of inequalities for this situation is:

\[
\begin{align*}
    x & \geq 0 \\
    y & \geq 0 \\
    80x + 160y & > 2000
\end{align*}
\]
(b) Now we can graph the inequalities. Each tick-mark on the y-axis represents 2 units. Each tick-mark on the x-axis represents 4 units. First I rewrite each of the inequalities with the = symbol.

\[ x = 0 \quad y = 0 \quad 80x + 160y = 2000 \]

Now, we can find the intercepts for the third equation.

We set \( y = 0 \) to find the \( x \)-intercept:
\[
80x + 160(0) = 2000 \\
80x + 0 = 2000 \\
x = 25
\]

We set \( x = 0 \) to find the \( y \)-intercept:
\[
80(0) + 160y = 2000 \\
0 + 160y = 2000 \\
y = 12.5
\]
Therefore, the two intercepts for the third equation are: (25,0) and (0,12.5).

Graph $y \geq 0$. The blue line, $y = 0$, is solid: Equality is included in $y \geq 0$. Because (0,1) makes the inequality true, we shade the half-plane containing (0,1) in green.
Graph $x \geq 0$. The red line, $x = 0$, is solid: Equality is included in $x \geq 0$. Because $(1,0)$ makes the inequality true, we shade the half-plane containing $(1,0)$ in yellow.
Graph $80x + 160y > 2000$. The brown line, $80x + 160y = 2000$, is dashed: Equality is not included in $80x + 160y > 2000$. Because $(0,0)$ makes the inequality false, we shade the half-plane not containing $(0,0)$ in gray.
The region that all three graphs have in common is the solution.

For any point in the shaded region where the $x$ of the point is number of children and $y$ is number of adults – the elevator would be overloaded. The point (6,10) is in that shaded region. You shouldn’t put 10 adults & 6 children on this elevator then!