Chapter 6

Exponents and Polynomials
§ 6.1

Adding and Subtracting Polynomials
A polynomial is a single term or the sum of two or more terms containing variables with whole number exponents.

Consider the polynomial: \(3x^4 - 2x^3 - 5x + 6\)

This polynomial contains four terms. It is customary to write the terms in order of descending powers of the variable. This is the standard form of a polynomial. Here are two other polynomials which are written in standard form.

\(5x^3 - 7x^2 + 2x - 8\)
\(8x^4 - 3x + 6\)
Polynomials

The **degree of a polynomial** is the greatest degree of any term of the polynomial. The degree of a term \(a x^n y^m\) is \((n + m)\)

and the **coefficient** of the term is \(a\). If there is exactly one term of greatest degree, it is called the **leading term**. Its coefficient is called the leading coefficient. Consider the polynomial:

\[
3x^4 - 2x^3 - 5x + 6
\]

3 is the leading coefficient. The degree is 4.
The Degree of $ax^n$

- If $a \neq 0$, the degree of $ax^n$ is $n$. The degree of a nonzero constant term is 0. The constant 0 has no defined degree.

$$5x^3 - 7x^2 + 2x - 8$$

Degree of nonzero constant: 0
The **degree of a polynomial** is the degree of its highest order term.

**Example:**

Degree 3 Polynomial: \(5x^3 - 7x^2 + 2x - 8\)

Degree 4 Polynomial: \(8x^4 - 3x + 6\)
Special Polynomials

• Monomial: A polynomial with one term.
• Binomial: A polynomial with two terms.
• Trinomial: A polynomial with three terms.

Example: \[8x^4 - 3x + 6\]

This is a 4\(^\text{th}\) degree trinomial.
Polynomials

The Degree of $ax^n$

If $a \neq 0$, the degree of $ax^n$ is $n$. The degree of a nonzero constant is 0. The constant 0 has no defined degree.

Adding Polynomials

Polynomials are added by removing the parentheses that surround each polynomial (if any) and then combining like terms.

Subtracting Polynomials

To subtract two polynomials, change the sign of every term of the second polynomial. Add this result to the first polynomial.
**Adding Polynomials**

- Polynomials are added by combining *like terms*.
- *Like terms* are terms containing exactly the same variables to the same powers.

**Example:**

\[ 4x^2 + 6x^2 = (4 + 6)x^2 = 10x^2 \]

These like terms both contain \(x\) to the second power.

Add the coefficients and keep the same variable factor, \(x^2\)
EXAMPLE

Add: \((-2x^3 + 8x^2 - 10x + 2) + (5x^3 + 4x^2 - 8x - 7)\)

\((-2x^3 + 5x^3) + (8x^2 + 4x^2) + (-10x - 8x) + (2 - 7)\) \hspace{1cm} \text{Group like terms.}

\(3x^3 + 12x^2 + (-18x) + (-5)\) \hspace{1cm} \text{Combine like terms.}

\(3x^3 + 12x^2 - 18x - 5\)
Adding Polynomials

**EXAMPLE**

Add: \( (7x^3 + 6x^2 - 11x + 13) + (9x^3 - 11x^2 + 7x - 17) \)

**SOLUTION**

\[
\begin{align*}
(7x^3 + 6x^2 - 11x + 13) + (9x^3 - 11x^2 + 7x - 17) &= -7x^3 + 6x^2 - 11x + 13 + 9x^3 - 11x^2 + 7x - 17 \\
&= -7x^3 + 19x^3 + 6x^2 - 11x^2 - 11x + 7x + 13 - 17 \\
&= 12x^3 - 5x^2 - 4x - 4
\end{align*}
\]

Remove parentheses
Rearrange terms so that like terms are adjacent
Combine like terms
EXAMPLE

Subtract: $(8x^2 - 10x + 2) - (5x^2 - 6x - 7)$

$(8x^2 - 10x + 2) + (-5x^2 + 6x + 7)$

Add the opposite of the polynomial being subtracted.

$(8x^2 - 5x^2) + (-10x + 6x) + (2 + 7)$

Group like terms.

$3x^2 + (-4x) + 9$

Combine like terms.

$3x^2 - 4x + 9$
Subtracting Polynomials

EXAMPLE

Subtract $\left(x^4 y^2 + 6x^3 y - 7y\right) - \left(x^4 y^2 - 5x^3 y - 6y + 8x\right)$

SOLUTION

$\left(x^4 y^2 + 6x^3 y - 7y\right) - \left(x^4 y^2 - 5x^3 y - 6y + 8x\right)$

$5x^4 y^2 + 6x^3 y - 7y - 3x^4 y^2 + 5x^3 y + 6y - 8x$

Change subtraction to addition and change the sign of every term of the polynomial in parentheses.

$\underbrace{5x^4 y^2 - 3x^4 y^2} + \underbrace{6x^3 y} + \underbrace{5x^3 y} - \underbrace{7y} + \underbrace{6y} - \underbrace{8x}$

Rearrange terms

$2x^4 y^2 + 11x^3 y - y - 8x$

Combine like terms
Polynomial Functions

\[ f(x) = 4x^3 - 2x^2 - 5x + 2 \]

is an example of a **polynomial function**. In a polynomial function, the expression that defines the function is a polynomial.

How do you evaluate a polynomial function? Use *Substitution*.
Polynomial functions of degree 2 or higher have graphs that are smooth and continuous.

By smooth, we mean that the graph contains only rounded corners with no sharp corners.

By continuous, we mean that the graph has no breaks and can be drawn without lifting the pencil from the page.
The graph below does not represent a polynomial function. Although it has a couple of smooth, rounded corners, it also has a sharp corner and a break in the graph. Either one of these last two features disqualifies it from being a polynomial function.
§ 6.2

Multiplying Polynomials
Product Rule

When multiplying exponential expressions with the same base, **add the exponents**.

\[ b^m b^n = b^{m+n} \]

A time to add...
Multiply $x^3 \cdot x^5$

$$x^3 x^5 = x^{3+5} = x^8$$

We used the Product Rule. **When multiplying exponential expressions with the same base, add the exponents.**

Note that the product rule does not apply to exponential expressions with different bases.
EXAMPLE

Multiply $2^3 \cdot 2^2$

$2^3 \cdot 2^2 = 2^{3+2} = 2^5$

We used the Product Rule. When multiplying exponential expressions with the same base, add the exponents. **Note:** We do not change the base!
The Power Rule

\[(b^m)^n = b^{mn}\]

When an exponential expression is raised to a power, multiply the exponents.

A time to multiply...
Find \((x^2)^5\)

\[
(x^2)^5 = x^{2(5)} = x^{10}
\]

We used the Power Rule. When an exponential expression is raised to a power, multiply the exponents.
EXAMPLE

Find \((3^4)^2\)

\[
(3^4)^2 = 3^{4(2)} = 3^8
\]

We used the Power Rule. When an exponential expression is raised to a power, multiply the exponents. \textbf{Note: We do not change the base!}
When a product is raised to a power, raise each factor to the power.

\[(ab)^n = a^n b^n\]
EXAMPLE

Find \((5x^3)^2\)

\[
(5x^3)^2 = 5^2(x^3)^2 \quad \text{Products to a Power Rule}
\]

\[
= 5^2x^6 \quad \text{Power Rule}
\]

\[
= 25x^6
\]

First, we used the Products to a Power Rule. When a product is raised to a power, raise each factor to the power.

Second, we used the Power Rule. When an exponential expression is raised to a power, multiply the exponents.
To multiply monomials with the same variable base, *multiply the coefficients* and then multiply the variable parts. *Use the product rule for exponents* to multiply the variables: Keep the variable and *add the exponents*.
Multiply the coefficients and multiply the variables.

Multiply (3x^2)(5x^7)

\[(3x^2)(5x^7) = (3 \cdot 5)(x^2x^7)\]

\[= 15x^{2+7}\]

Add the exponents.

\[= 15x^9\]

Simplify.
To multiply a monomial and a polynomial that is not a monomial, use the **distributive property** to multiply each term of the polynomial by the monomial.
Multiply the coefficients and add exponents.

\[ 3x(2x^2 - 5x + 7) \]

\[ = 3x(2x^2) - 3x(5x) + 3x(7) \]

\[ = 3 \cdot 2x^{1+2} - 3 \cdot 5x^{1+1} + 3 \cdot 7x \]

\[ = 6x^3 - 15x^2 + 21x \]
Multiply each term of one polynomial by each term of the other polynomial. Then combine like terms.

For example, if multiplying a binomial by a trinomial – you would have 6 products initially before you combined like terms.
Multiply $(3x + 2)(2x - 7)$

$$(3x + 2)(2x - 7) = 3x(2x - 7) + 2(2x - 7)$$

$$= 3x(2x) - 3x(7) + 2(2x) - 2(7)$$

$$= 6x^2 - 21x + 4x - 14$$

$$= 6x^2 - 17x - 14$$
Multiplying Polynomials

**EXAMPLE**

Multiply 

\[(x^4 y^2)(x^7 y)\]

**SOLUTION**

\[(x^4 y^2)(x^7 y)\]

\[6 \cdot 3 \cdot x^4 \cdot x^7 \cdot y^2 \cdot y^1\]

\[18x^{4+7} \cdot y^{2+1}\]

\[18x^{11} \cdot y^3\]

Rearrange factors

Multiply coefficients and add exponents

Simplify
EXAMPLE

Multiply $6x^3 (x^5 - 5x^2 + 7)$

SOLUTION

$6x^3 \cdot 3x^5 - 6x^3 \cdot 5x^2 + 6x^3 \cdot 7$

$18x^8 - 30x^5 + 42x^3$

Distribute

Multiply coefficients and add exponents
EXAMPLE

Multiply \( (a + b)(a^2 - ab + b^2) \).

SOLUTION

\[
(a + b)(a^2 - ab + b^2) = a \cdot a^2 - a \cdot ab + a \cdot b^2 + b \cdot a^2 - b \cdot ab + b \cdot b^2 \\
n = a^3 - a^2 b + ab^2 + a^2 b - ab^2 + b^3 \\
= a^3 + b^3
\]

Note this: We multiply each term of the binomial by each term of the trinomial. We get 6 products in all.
§ 6.3

Special Products
In this section we will use the **distributive property** to develop patterns that can help you in multiplying some special binomials quickly.

We will find the product of two binomials using a method called **FOIL**. You will be thinking…”first two, outer two, inner two, last two” before the section is over.

We will learn a formula for finding the square of a binomial sum. You will learn a formula for finding the product of the sum and difference of two terms.

And whether you choose to take a handy shortcut and use these formulas or simply use polynomial multiplication will be left to you to decide.
Multiplying Polynomials - FOIL

Using the FOIL Method to Multiply Binomials

\[(a + b)(c + d) = ac + ad + bc + bd\]

- **F**irst terms: \(ac\)
- **O**utside terms: \(ad\)
- **I**nside terms: \(bc\)
- **L**ast terms: \(bd\)

**Diagram:**

- **First** (F)
- **Outside** (O)
- **Inside** (I)
- **Last** (L)

Product of **First** terms: \(ac\)
Product of **Outside** terms: \(ad\)
Product of **Inside** terms: \(bc\)
Product of **Last** terms: \(bd\)
The FOIL Method

Product of Two Binomials

Distribute each term in the first binomial through each term of the second binomial.

\[(ax + b)(cx + d) = ax \cdot cx + ax \cdot d + b \cdot cx + b \cdot d\]
**EXAMPLE**

Multiply \((4x + 3)(6x + 1)\).

**SOLUTION**

\[
(4x + 3)(6x + 1) = 4x \cdot 6x + 4x \cdot 1 + 3 \cdot 6x + 3 \cdot 1
\]

= \(20x^2 + 4x + 15x + 3\)

= \(20x^2 + 19x + 3\)

Multiply

Combine like terms
Multiply \((5x + 2)(x + 7)\)

\[
(5x + 2)(x + 7) = 5x \cdot x + 5x \cdot 7 + 2 \cdot x + 2 \cdot 7
\]

\[
= 5x^2 + 35x + 2x + 14
\]

\[
= 5x^2 + 37x + 14
\]
EXAMPLE

Multiply \((3x - 5)(2x - 8)\)

\[
(3x - 5)(2x - 8) = 3x \cdot 2x + 3x \cdot (-8) + (-5) \cdot 2x + (-5) \cdot (-8)
\]

\[
= 5x^2 + 35x + 2x + 14
\]

\[
= 5x^2 + 37x + 14
\]

First two, Outer two, Inner two, Last two...
The Square of a Binomial Sum

\[(A + B)^2 = A^2 + 2AB + B^2\]

The square of a binomial sum is the first term squared plus two times the product of the terms plus the last term squared.
(A + B)(A − B) = A^2 - B^2

The product of the sum and the difference of the same two terms is the square of the first term minus the square of the second term.
EXAMPLE

Multiply \((x - 5)(x + 5)\)

Since this is the product of a sum and a difference, we use the rule:

\[
(A + B)(A - B) = A^2 - B^2
\]

\[
(x - 5)(x + 5) = x^2 - 5^2
\]

\[
= x^2 - 25
\]
EXAMPLE

Find \((x + 7)^2\)

Since this is the square of a binomial sum, we use the rule:

\[(A + B)^2 = A^2 + 2AB + B^2\]

\[(x + 7)^2 = x^2 + 2x(7) + 7^2\]

\[= x^2 + 14x + 49\]
\[(A - B)^2 = A^2 - 2AB + B^2\]

The square of a binomial difference is the first term squared minus two times the product of the terms plus the last term squared.
EXAMPLE

Find \((3x - 5)^2\)

Since this is the square of a binomial difference, we use the rule:

\[(A - B)^2 = A^2 - 2AB + B^2\]

\[(3x - 5)^2 = (3x)^2 - 2 \cdot 3x(5) + 5^2\]

\[= 9x^2 - 30x + 25\]
EXAMPLE

Find \((x + 7)^2\)

Since this is the square of a binominal sum, we use the rule:

\[(A + B)^2 = A^2 + 2AB + B^2\]

\[(x + 7)^2 = x^2 + 2x(7) + 7^2\]

\[= x^2 + 14x + 49\]
EXAMPLE

Find \((3x - 5)^2\)

Since this is the square of a binomial difference, we use the rule:

\[(A - B)^2 = A^2 - 2AB + B^2\]

\[(3x - 5)^2 = (3x)^2 - 2 \cdot 3x(5) + 5^2\]

\[= 9x^2 - 30x + 25\]
<table>
<thead>
<tr>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>The Square of a Binomial Sum</strong></td>
</tr>
<tr>
<td>$(A + B)^2 = A^2 + 2AB + B^2$</td>
</tr>
<tr>
<td><strong>The Square of a Binomial Difference</strong></td>
</tr>
<tr>
<td>$(A - B)^2 = A^2 - 2AB + B^2$</td>
</tr>
<tr>
<td><strong>The Product of the Sum and Difference of Two Terms</strong></td>
</tr>
<tr>
<td>$(A + B)(A - B) = A^2 - B^2$</td>
</tr>
</tbody>
</table>
### EXAMPLE

Multiply \((4x + y)^2\).

### SOLUTION

Use the special-product formula shown.

\[
A + B \overset{2}{=} A^2 + 2AB + B^2
\]

<table>
<thead>
<tr>
<th>Term</th>
<th>First Term (^2)</th>
<th>2 · Product of the Terms</th>
<th>Last Term (^2)</th>
<th>= Product</th>
</tr>
</thead>
<tbody>
<tr>
<td>((4x + y)^2)</td>
<td>((4x)^2)</td>
<td>2 · (4x \cdot y)</td>
<td>(y^2)</td>
<td>(16x^2 + 8xy + y^2)</td>
</tr>
</tbody>
</table>
EXAMPLE

Multiply \((x - 4y)^2\).

SOLUTION

Use the special-product formula shown.

\[
\begin{align*}
A - B^2 &= A^2 - 2AB + B^2 \\
\text{First Term}^2 &- \text{Product of the Terms} + \text{Last Term}^2 = \text{Product}
\end{align*}
\]

\[
(\quad x \quad)^2 - 2 \cdot 3x \cdot 4y + \quad (\quad y \quad)^2 = 9x^2 - 24xy + 16y^2
\]
EXAMPLE

Multiply \((xy^2 - 4y)(xy^2 + 4y)\).

SOLUTION

Use the special-product formula shown.

\[
(A + B)(A - B) = A^2 - B^2
\]

First Term Squared  -  Second Term Squared  =  Product

\[
= (xy^2)^2 - (4y)^2 = 9x^2y^4 - 16y^2
\]
§ 6.4

Polynomials in Several Variables
A polynomial containing two or more variables is called a polynomial in several variables. An example of a polynomial in two variables is:

$$5x^3y + 6xy - 2x^2y$$
1. Substitute the given value for each variable.

2. Perform the resulting computation using the order of operations.
EXAMPLE

Evaluate $5x^3y + 6xy - 2x^2y$ for $x = 3$ and $y = -1$.

1. Substitute the given value for each variable.

   $$5(3)^3(-1) + 6(3)(-1) - 2(3)^2(-1)$$

2. Perform the resulting computation using the order of operations.

   $$5(27)(-1) + 6(3)(-1) - 2(9)(-1)$$

   $$-135 - 18 + 18$$

   $$-135$$
• Polynomials in several variables are added by combining like terms.

• Polynomials in several variables are subtracted by adding the first polynomial and the opposite of the second polynomial.

❖ *Like terms* are terms containing exactly the same variables to the same powers.
EXAMPLE

Add: $(2x^3y - 8xy^2 + 3xy) + (5x^3y + 4xy^2 + 7xy)$

$$= (2x^3y + 5x^3y) + (-8x^2y + 4x^2y) + (3xy + 7xy)$$  \[\text{Group like terms.}\]

$$= 7x^3y - 4x^2y + 10xy$$  \[\text{Combine like terms.}\]
Multiply coefficients and add exponents on variables with the same base.

\[(4x^3y^2)(7xy^5)\]

\[= (4 \cdot 7)(x^3x)(y^2y^5)\] \hspace{1cm} \text{Regroup.}

\[= 28x^4y^7\] \hspace{1cm} \text{Multiply the coefficients and add the exponents.}
EXAMPLE

Multiply each term of the polynomial by the monomial.

\[(4x^3y)(2x^4y^2 + 7xy - 2)\]

\[
= (4x^3y)(2x^4y^2) + (4x^3y)(7xy) - (4x^3y)2
\]

\[
= 8x^7y^3 + 28x^4y^2 - 8x^3y
\]
EXAMPLE

Multiply each term of one polynomial by each term in the other polynomial. (For Binomial \cdot Binomial use FOIL.)

\[(2x^3y - 4)(3xy + 5)\]

\[= (2x^3y)(3xy) + (2x^3y)(5) - (4)(3xy) - 4(5)\]

\[= 6x^4y^2 + 10x^3y - 12xy - 20\]

Use the distributive property.

Multiply the coefficients and add the exponents.
## Polynomials

### EXAMPLE

Determine the coefficient of each term, the degree of each term, the degree of the polynomial, the leading term, and the leading coefficient of the polynomial.

\[ 12x^4y - 5x^3y^7 - x^2 + 4 \]

### SOLUTION

<table>
<thead>
<tr>
<th>Term</th>
<th>Coefficient</th>
<th>Degree (Sum of Exponents on the Variables)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 12x^4y )</td>
<td>12</td>
<td>( 4 + 1 = 5 )</td>
</tr>
<tr>
<td>( -5x^3y^7 )</td>
<td>-5</td>
<td>( 3 + 7 = 10 )</td>
</tr>
<tr>
<td>( -x^2 )</td>
<td>-1</td>
<td>( 2 + 0 = 2 )</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>( 0 + 0 = 0 )</td>
</tr>
</tbody>
</table>
CONTINUED

\[12x^4 y - 5x^3 y^7 - x^2 + 4\]

The degree of the polynomial is the greatest degree of all its terms, which is 10. The leading term is the term of the greatest degree, which is \(-5x^3 y^7\). Its coefficient, -5, is the leading coefficient.
Subtracting Polynomials

**EXAMPLE**

Subtract \( (x^4y^2 + 6x^3y - 7y) - (x^4y^2 - 5x^3y - 6y + 8x) \).

**SOLUTION**

\[
(x^4y^2 + 6x^3y - 7y) - (x^4y^2 - 5x^3y - 6y + 8x) = 5x^4y^2 + 6x^3y - 7y - 3x^4y^2 + 5x^3y + 6y - 8x
\]

Change subtraction to addition and change the sign of every term of the polynomial in parentheses.

\[
5x^4y^2 - 3x^4y^2 + 6x^3y + 5x^3y - 7y + 6y - 8x
\]

Rearrange terms

\[
2x^4y^2 + 11x^3y - y - 8x
\]

Combine like terms.
§ 6.5

Dividing Polynomials
When dividing exponential expressions with the same nonzero base, subtract the exponent in the denominator from the exponent in the numerator. Use this difference as the exponent on the common base.
EXAMPLE

Divide: \( \frac{x^7}{x^2} \)

\[ = x^{7-2} \]

\[ = x^5 \]
EXAMPLE

Divide: \( \frac{5^4}{5} \)

\[ = 5^{4-1} \]

\[ = 5^3 \text{ or } 125 \]
The Quotient Rule for Exponents

**EXAMPLE**

Divide: \( \frac{y^7}{y^7} \)

\[ = y^{7-7} \]

\[ = y^0 \]

But we know any nonzero expression divided by itself is 1.

So \( \frac{y^7}{y^7} = 1 = y^0 \)
If $b$ is any real number other than 0,

$$b^0 = 1$$
Zero as an Exponent

**EXAMPLES**

\[ 5^0 = 1 \quad \text{5 is raised to the 0 power.} \]

\[ (2xy)^0 = 1 \quad \text{2xy is raised to the 0 power.} \]

\[ 2xy^0 = 2x \quad \text{Only y is raised to the 0 power.} \]

\[ -2^0 = -1 \quad \text{Only 2 is raised to the 0 power.} \]
If $a$ and $b$ are real numbers and $b$ is nonzero, then

$$
\left( \frac{a}{b} \right)^n = \frac{a^n}{b^n}
$$

When a quotient is raised to a power, raise the numerator to the power and divide by the denominator raised to the power.
EXAMPLES

\[
\left( \frac{2}{x} \right)^3 = \frac{2^3}{x^3} = \frac{8}{x^3}
\]

Cube the numerator and denominator.

\[
\left( \frac{a^3}{b^5} \right)^2 = \frac{(a^3)^2}{(b^5)^2} = \frac{a^6}{b^{10}}
\]

Square the numerator and denominator.
EXAMPLE

\[
\left( \frac{5x^4}{y^2} \right)^3
\]

\[
= \frac{(5x^4)^3}{(y^2)^3}
\]

Cube the numerator and denominator.

\[
= \frac{5^3(x^4)^3}{(y^2)^3}
\]

Cube each factor in the numerator.

\[
= \frac{125x^{12}}{y^6}
\]

Simplify.
To divide monomials, *divide the coefficients* and then *divide the variables (by subtracting exponents)*. Use the quotient rule for exponents to divide the variable factors. Keep the variable and subtract the exponents.
Now we will look at dividing a polynomial by a monomial.

Division of a polynomial by a monomial is relatively easy—you just divide each term of the polynomial by the monomial. The number of separate divisions you will have is the number of terms in the polynomial.
Dividing Monomials

EXAMPLE

Divide: \( \frac{5x^7}{10x^3} \)

\[
\begin{align*}
&= \frac{1x^{7-3}}{2} \\
&= \frac{x^4}{2} \\
&= \frac{x^4}{2}
\end{align*}
\]

Divide the coefficients, 5/10 = 1/2, then divide the variables by subtracting exponents. Simplify.
Dividing Monomials

EXAMPLE

\[
\frac{6x^5y^3}{2x^3y}
\]

= 3\(x^{5-3}y^{3-1}\)  \[\text{Divide the coefficients, } 6/2 = 3, \text{ then divide the variables by subtracting exponents.}\]

= 3\(x^2y^2\)  \[\text{Simplify.}\]
Dividing a Polynomial by a Monomial

To divide a polynomial by a monomial, divide each term of the polynomial by the monomial.
Division of Polynomials

EXAMPLE

Divide: \( \frac{40x^4 y^3 - 20x^3 y^2 - 50x^2 y}{10x^3 y^3} \)

SOLUTION

Express the division in a vertical format.

Divide each term of the polynomial by the monomial. Note the 3 separate quotients.

Simplify each quotient.
Dividing a Polynomial by a Monomial

Divide: \( \frac{5x^6 + 2x^4 - 10x^2}{10x} \)

\[
= \frac{5x^6}{10x} + \frac{2x^4}{10x} - \frac{10x^2}{10x}
\]

Divide each term of the polynomial by the monomial.

\[
= \frac{1x^{6-1}}{2} + \frac{1x^{4-1}}{5} - \frac{x^{2-1}}{1}
\]

Divide the coefficients, then divide the variables by subtracting exponents.

\[
= \frac{x^5}{2} + \frac{2x^3}{5} - x
\]

Simplify.
§ 6.6

Dividing Polynomials by Binomials
In the last section we looked at dividing by a monomial. In this section we will look at dividing by a binomial.

Division of a polynomial by a monomial was a relatively easy task as we saw – we just divided each term of the polynomial by the monomial. The number of separate divisions we had was the number of terms in the polynomial.

The second case, that of dividing a polynomial by a binomial or any other polynomial having more than one term, is more difficult. This requires a process of long division.
Division of Polynomials

We will now consider the harder problem – that of dividing a polynomial by a binomial.

The four steps that you remember using in long division of whole numbers – divide, multiply, subtract, bring down the next term – form the same repetitive procedure for polynomial long division.

Carefully consider and try to remember the four terms illustrated on the next slide. These terms are: quotient, divisor, dividend, and remainder.
Division of Polynomials

EXAMPLE

\[
\begin{array}{c}
\text{DIVISOR} \\
\overset{x + 2}{4x + 7} \\
\text{QUOTIENT} \\
\text{DIVIDEND} \\
\text{REMAINDER}
\end{array}
\]

\[
x + 2) \quad 4x + 7 \\
\underline{4x + 8} \\
-1
\]
EXAMPLE

Divide: \( \left( x^3 + 5x^2 + 7x + 2 \right) \div \left( x + 2 \right) \)

SOLUTION

Arrange the terms of the dividend, \( x^3 + 5x^2 + 7x + 2 \), and the divisor, \( x + 2 \), in descending powers of \( x \).

\[
\begin{array}{c}
\begin{array}{c}
\text{Divide } x^3 \text{ (the first term in the dividend) by } x \text{ (the first term in the divisor). Align like terms.}
\end{array}
\end{array}
\]
Division of Polynomials

CONTINUED

\[
x + 2 \overbrace{\quad \quad}^{x^2} x^3 + 5x^2 + 7x + 2
\]

\[
x^3 + 2x^2
\]

Multiply each term in the divisor \((x + 2)\) by \(x^2\), aligning terms of the product under like terms in the dividend.

\[
x + 2 \overbrace{\quad \quad}^{x^2} x^3 + 5x^2 + 7x + 2
\]

\[
x^3 + 2x^2
\]

\[
3x^2
\]

Subtract \(x^3 + 2x^2\) from \(x^3 + 5x^2\) by changing the sign of each term in the lower expression and then adding.
Division of Polynomials

CONTINUED

\[
\begin{array}{c}
x^2 \\
x + 2 \overline{x^3 + 5x^2 + 7x + 2} \\
x^3 + 2x^2 \\
3x^2 + 7x \\
\end{array}
\]

Bring down 7x from the original dividend and add algebraically to form a new dividend.

\[
\begin{array}{c}
3x^2 + 7x \\
\end{array}
\]

Find the second term of the quotient. Divide the first term of \(3x^2 + 7x\) by \(x\), the first term of the divisor.
Multiply the divisor \((x + 2)\) by 3\(x\), aligning under like terms in the new dividend. Then subtract.

\[
\begin{array}{c}
x^2 + 3x \\
\hline
x + 2) x^3 + 5x^2 + 7x + 2 \\
x^3 + 2x^2 \\
\hline
3x^2 + 7x \\
x^2 + 6x \\
\hline
x
\end{array}
\]
CONTINUED

\[
x + 2 \overline{\begin{array}{c}
x^3 + 5x^2 + 7x + 2 \\
x^3 + 2x^2 \\
\hline
\end{array}}
\]

\[
3x^2 + 7x
\]

\[
3x^2 + 6x
\]

\[
x + 2
\]

\[
x + 2
\]

\[
0
\]

Bring down 2 from the original dividend and add algebraically to form a new dividend.

Find the third term of the quotient, 1. Divide the first term of \(x + 2\) by \(x\), the first term of the divisor.

Multiply the divisor by 1, aligning under like terms in the new dividend. Then subtract to obtain the remainder of 0.
The quotient is \( x^2 + 3x + 1 \) and the remainder is 0. We will not list a remainder of 0 in the answer. Thus,

\[
x^3 + 5x^2 + 7x + 2 \div x + 2 = x^2 + 3x + 1.
\]
### Long Division of Polynomials

1) **Arrange the terms** of both the dividend and the divisor in descending powers of any variable.

2) **Divide** the first term in the dividend by the first term in the divisor. The result is the first term of the quotient.

3) **Multiply** every term in the divisor by the first term in the quotient. Write the resulting product beneath the dividend with like terms lined up.

4) **Subtract** the product from the dividend.

5) **Bring down** the next term in the original dividend and write it next to the remainder to form a new dividend.

6) Use this new expression as the dividend and repeat this process until the remainder can no longer be divided. This will occur when the degree of the remainder (the highest exponent on a variable in the remainder) is less than the degree of the divisor.
EXAMPLE

Divide: \( (x^5 - x^3 + 4x^2 - 12x - 8) \div (x^2 - 2) \)

SOLUTION

We write the dividend, \( 3x^5 - x^3 + 4x^2 - 12x - 8 \), as \( 3x^5 + 0x^4 - x^3 + 4x^2 - 12x - 8 \) to keep all like terms aligned. For the same reason, we write the divisor, \( x^2 - 2 \), as \( x^2 + 0x - 2 \).

Note that when terms are missing in the dividend, you should insert the term using a coefficient of 0. This is to keep the terms aligned. The term with the 0 coefficient is still equal to 0, but that term serves as an effective placeholder.
CONTINUED

\[
x^2 + 0x - 2 \div \left( \frac{3x^3}{3x^5 + 0x^4 - x^3 + 4x^2 - 12x - 8} \right)
\]

\[
\begin{align*}
3x^3 & \quad + 5x + 4 \\
3x^5 & + 0x^4 - 6x^3 \\
5x^3 & + 4x^2 - 12x \\
5x^3 & + 0x^2 - 10x \\
4x^2 & - 2x - 8 \\
4x^2 & + 0x - 8 \\
\end{align*}
\]

\[
-2x
\]
CONTINUED

The division process is finished because the degree of \(-2x\), which is 1, is less than the degree of the divisor \(x^2 - 2\), which is 2. The answer is

\[
\frac{3x^5 - x^3 + 4x^2 - 12x - 8}{x^2 - 2} = 3x^3 + 5x + 4 + \frac{-2x}{x^2 - 2}.
\]
Important to Remember:

To divide by a polynomial containing more than one term, use long division. If necessary, arrange the dividend in descending powers of the variable. Do the same with the divisor. If a power of a variable is missing in the dividend, add that term using a coefficient of 0.

Repeat the four steps of the long-division process – divide, multiply, subtract, bring down the next term – until the degree of the remainder is less than the degree of the divisor.

When the degree of the remainder is less than the degree of the divisor – you know you are done!
§ 6.7

Negative Exponents and Scientific Notation
We frequently encounter very large or very small numbers. Think about the size of the national debt (BIG!) or the diameter of an atom (small!). In this section we use exponents to put really big or really small numbers into perspective.

We will first define **negative exponents** and then will use these for writing numbers in **scientific notation**.

We begin by reviewing our exponent rules.
## Properties of Exponents

<table>
<thead>
<tr>
<th>Exponent Rules</th>
<th>Formula</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Product Rule</strong></td>
<td>$b^m \times b^n = b^{m+n}$</td>
<td>When multiplying exponential expressions with the same base, add the exponents.</td>
</tr>
<tr>
<td><strong>Quotient Rule</strong></td>
<td>$\frac{b^m}{b^n} = b^{m-n}, b \neq 0$</td>
<td>When dividing exponential expressions with the same nonzero base, subtract the exponent in the denominator from the exponent in the numerator.</td>
</tr>
</tbody>
</table>
# Properties of Exponents

<table>
<thead>
<tr>
<th>Exponent Rules</th>
<th>Examples</th>
</tr>
</thead>
</table>
| **Product Rule** | \[q^3 q^5 q^2 = q^{3+5+2} = q^{10}\]  
| | \[a^2 b^3 c^6 \cdot a^4 c^7 = 4 \cdot 3 \cdot a^2 \cdot a^4 \cdot b^3 \cdot c^6 \cdot c^7 = 12 a^{2+4} b^3 c^{6+7} = 12 a^6 b^3 c^{13}\] |
| **Quotient Rule** | \[\frac{z^8}{z^3} = z^{8-3} = z^5\]  
| | \[\frac{18r^5 q^3 t^6}{9r^3 q^2 t^4} = \frac{18}{9} \cdot \frac{r^5}{r^3} \cdot \frac{q^3}{q^2} \cdot \frac{t^6}{t^4} = 2r^{5-3} q^{3-2} t^{6-4} = 2r^2 q t^2\] |
Properties of Exponents

The Zero Exponent Rule: If $b$ is any real number other than 0, then $b^0 = 1$

Negative Exponent Rule: If $b$ is any real number other than 0 and $n$ is a natural number, then

$$b^{-n} = \frac{1}{b^n} \quad \text{and} \quad \frac{1}{b^{-n}} = b^n$$
Write $x^{-4}$ with positive exponents.

$$= \frac{1}{x^4}.$$ 

Write $5^{-3}$ with positive exponents.

$$= \frac{1}{5^3}.$$
If $b$ is any real number other than 0 and $n$ is a natural number, then

\[ b^{-n} = \frac{1}{b^n} \quad \text{and} \quad \frac{1}{b^{-n}} = b^n \]

When a negative number appears as an exponent, switch the position of the base (from numerator to denominator or denominator to numerator) and make the exponent positive. *The sign of the base does not change.*
Write with positive exponents.

\[
\frac{2y^{-3}}{5x^{-2}}
\]

When a negative number appears as an exponent, switch the position of the base. Here, the \(y^{-3}\) moves from numerator to denominator as \(y^3\) and the \(x^{-2}\) moves from the denominator to numerator as \(x^2\) The sign of the base does not change.

\[
\frac{2x^2}{5y^3}
\]
## Properties of Exponents

<table>
<thead>
<tr>
<th>Exponent Rules</th>
<th>Examples</th>
</tr>
</thead>
<tbody>
<tr>
<td>Zero Exponent Rule</td>
<td>(17^0 = 1)</td>
</tr>
<tr>
<td></td>
<td>(x^3 y^2 z^{34} - 1^0 = 1)</td>
</tr>
<tr>
<td>Negative Exponent Rule</td>
<td>(q^{2} = \frac{1}{q^2})</td>
</tr>
<tr>
<td></td>
<td>(13P^{-3}Q^4R^{-5} = 13 \cdot \frac{1}{P^3} \cdot Q^4 \cdot \frac{1}{R^5} = \frac{13Q^4}{P^3R^5})</td>
</tr>
</tbody>
</table>
## Properties of Exponents

<table>
<thead>
<tr>
<th>Exponent Rules</th>
<th>Examples</th>
</tr>
</thead>
</table>
| Negative Exponents in Numerators and Denominators | \[
\frac{4x^{-3} y^6}{\phi q^{-2}} = \frac{4 y^6 \phi q^2}{x^3}
\]
|                                | \[
\frac{2}{3^{-2} 4^{-3}} = 2 \cdot 3^2 \cdot 4^3 = 2 \cdot 3 \cdot 3 \cdot 4 \cdot 4 \cdot 4 = 1152
\] |
|                                | \[
\phi x^{\phi} = \phi^{4x}
\]
|                                | \[
\phi^{-2} x^{-3} = d^{-2 \cdot 3} = d^{-6} = \frac{1}{d^6}
\] |
### Properties of Exponents

<table>
<thead>
<tr>
<th>Exponent Rules</th>
<th>Examples</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Products to Powers</strong></td>
<td>$x^3 y^2 = 4^4 \cdot (x^2 y^4) = 256 \cdot x^{3+4} \cdot y^{2+4} = 256x^{12}y^8$</td>
</tr>
<tr>
<td></td>
<td>$5a^{-4}b^6 = 5 \cdot 2^{-3} \cdot (a^{-4} \cdot b^6) = 5 \cdot 2^{-3} \cdot a^{-4-3} \cdot b^{6-3} =$</td>
</tr>
<tr>
<td></td>
<td>$5 \cdot 2^{-3} \cdot a^{12} \cdot b^{-18} = 5 \cdot \frac{1}{2^3} \cdot a^{12} \cdot \frac{1}{b^{18}} = \frac{5a^{12}}{8b^{18}}$</td>
</tr>
<tr>
<td><strong>Quotients to Powers</strong></td>
<td>$\left(\frac{a^{-2}}{b^{-3}}\right)^4 = \left(\frac{a^{-2}}{b^{-3}}\right)^4 = \frac{a^{-2-4}}{b^{-3-4}} = \frac{a^8}{b^{12}}$</td>
</tr>
<tr>
<td></td>
<td>$\left(\frac{3g^3h^{-2}}{7f}\right)^2 = \left(\frac{3g^3}{7fh^2}\right)^2 = \frac{3^2 \cdot (g^3)^2}{7^2 \cdot f^2 \cdot (h^2)^2} = \frac{9g^{3\cdot2}}{49f^2h^{2\cdot2}} = \frac{9g^6}{49f^2h^4}$</td>
</tr>
</tbody>
</table>
## Simplifying Exponential Expressions

<table>
<thead>
<tr>
<th>Simplification Techniques</th>
<th>Examples</th>
</tr>
</thead>
<tbody>
<tr>
<td>If necessary, remove parentheses by using the Products to Powers Rule or the Quotient to Powers Rule.</td>
<td>$a^b \cdot a^c = a^{b+c}$</td>
</tr>
<tr>
<td></td>
<td>$(\frac{17}{x^2})^3 = \frac{17^3}{x^{2\cdot3}} = \frac{4913}{x^6}$</td>
</tr>
<tr>
<td>If necessary, simplify powers to powers by using the Power Rule.</td>
<td>$Q^{71\cdot19} = Q^{1349}$</td>
</tr>
<tr>
<td></td>
<td>$W^{5\cdot10} = W^{50}$</td>
</tr>
</tbody>
</table>
## Simplifying Exponential Expressions

<table>
<thead>
<tr>
<th>Simplification Techniques</th>
<th>Examples</th>
</tr>
</thead>
<tbody>
<tr>
<td>Be sure each base appears only once in the final form by using the Product Rule or Quotient Rule</td>
<td>$H^4 \cdot H^{16} = H^{4+16} = H^{20}$</td>
</tr>
<tr>
<td>If necessary, rewrite exponential expressions with zero powers as 1. Furthermore, write the answer with positive exponents by using the Negative Exponent Rule</td>
<td>$\frac{V^{23}}{V^{17}} = V^{23-17} = V^6$</td>
</tr>
<tr>
<td></td>
<td>$3 + 2 \left(5X^3Y^{-4}\right) = 3 + 2 \cdot 1 = 5$</td>
</tr>
<tr>
<td></td>
<td>$\frac{31}{K^{-12}} = 31K^{12}$</td>
</tr>
</tbody>
</table>
Simplifying Exponential Expressions

Of importance to note…

An exponential expression is “simplified” when

Each base occurs only once.

No parentheses appear.

No powers are raised to powers.

No negative or zero exponents appear.
When a negative number appears as an exponent, switch the position of the base. The $x^{-2}$ moves from numerator to denominator as $x^2$. 

Simplify $\frac{(2x^4)^3}{5x^8}$

EXAMPLE

$= \frac{2^3(x^2)^3}{5x^8}$

Cube each factor in the numerator.

$= \frac{8x^6}{5x^8}$

Multiply powers using $(b^m)^n = b^{mn}$.

$= \frac{8x^{-2}}{5}$

Division with the same base, subtract exponents.

$= \frac{8}{5x^2}$

When a negative number appears as an exponent, switch the position of the base. The $x^{-2}$ moves from numerator to denominator as $x^2$. 

Blitzer, Introductory Algebra, 5e – Slide #112 Section 6.7
Properties of Exponents

Of importance to note…

Be aware that a negative exponent does not make the value of the expression negative. **The sign of the exponent in no way affects the sign of the term.** The negative exponent, if it could talk, would simply be saying: “Take the reciprocal.”
Scientific Notation

At times you may find it necessary to work with really large numbers, or alternately, really small numbers. In this section, you will learn how to write these often cumbersome numbers in *scientific notation*.

A number is written in scientific notation when it is expressed as the product of a number between one and ten and some power of ten.
Scientific Notation

Study Tip:

“Big nonnegative numbers” have **positive** powers of ten when written in scientific notation. That is, if the absolute value of a number is BIG (greater than 10), it will have a positive exponent in scientific notation.

Small nonnegative numbers have **negative** powers of ten when written in scientific notation. That is, if the absolute value of a number is small (less than 1), it will have a negative exponent in scientific notation.
Converting from Decimal to Scientific Notation

- Determine $a$, the numerical factor. Move the decimal point in the given number to obtain a number greater than or equal to 1 and less than 10.
- Determine $n$, the exponent on $10^n$. The absolute value of $n$ is the number of places the decimal was moved. The exponent $n$ is positive if the given number is greater than 10 and negative if the given number is between 0 and 1.
## Scientific Notation

### Converting from Decimal to Scientific Notation

(Write the number in the form $a \times 10^n$)

1) Determine $a$, the numerical factor. Move the decimal point in the given number to obtain a number whose absolute value is between 1 and 10, including 1.

2) Determine $n$, the exponent on $10^n$. The absolute value of $n$ is the number of places the decimal point was moved. The exponent $n$ is positive if the decimal point was moved to the left, negative is the decimal point was moved to the right, and 0 if the decimal point was not moved.
Scientific Notation to Decimal Notation

**EXAMPLE**

Write each number in decimal notation.

\[ 3.45 \times 10^{-3} \quad -3.891 \times 10^8 \quad -5.94874 \times 10^{-4} \]

**SOLUTION**

\[ 3.45 \times 10^{-3} = 0.00345 \]

\[ -3.891 \times 10^8 = -389,100,000 \]

\[ -5.94874 \times 10^{-4} = -0.000594874 \]
EXAMPLE

Write each number in scientific notation.

324,510,000,000,000,000
0.0000000859
– 4395

SOLUTION

324,510,000,000,000,000 = 3.2451 \times 10^{17}

0.0000000859 = 8.59 \times 10^{-8}

– 4395 = −4.395 \times 10^3
**EXAMPLE**

Perform the indicated computation, writing the answer in scientific notation.

\[ 8 \times 10^2 \cdot 3 \times 10^4 = 2.3 \times 10^6 \]

**SOLUTION**

\[ 8 \times 10^2 \cdot 3 \times 10^4 \]

Regroup factors

Multiply

Simplify

Rewrite in scientific notation

\[ 2.3 \times 10^6 \]
Scientific Notation

EXAMPLE

Perform the indicated computation, writing the answer in scientific notation.

\[-3.2 \times 10^{-8} \div 5.41 \times 10^{-6}\]

SOLUTION

\[
\frac{-3.2 \times 10^{-8}}{5.41 \times 10^{-6}} = \frac{-3.2}{5.41} \times \frac{10^{-8}}{10^{-6}} = \frac{-3.2}{5.41} \times 10^{-8-6} = \frac{-3.2}{5.41} \times 10^{-2} = \frac{-0.59}{1} \times 10^{-3}
\]

Group factors
Divide
Simplify
Write in scientific notation
EXAMPLE

The area of Alaska is approximately $3.66 \times 10^8$ acres. The state was purchased in 1867 from Russia for $7.2$ million. What price per acre, to the nearest cent, did the United States pay Russia?

SOLUTION

We will first write $7.2$ million in scientific notation. It is:

$$7.2 \times 10^6$$

Now we can answer the question.
CONTINUED

\[
\frac{7.2 \times 10^6}{3.6 \times 10^8}
\]

Divide ‘dollar amount’ by ‘acreage’

\[
\frac{7.2 \times 10^6}{3.66 \times 10^8}
\]

Simplify

\[
1.97 \times 10^{6-8}
\]

Divide

Subtract

Therefore, since \(1.97 \times 10^{-2}\) dollars/acre equals $0.0197/acre, then the price per acre for Alaska was approximately 2 cents per acre. WOW!!!