Graphs, Linear Equations, and Functions

1. Interpret a line graph.
2. Plot ordered pairs.
3. Find ordered pairs that satisfy a given equation.
4. Graph lines.
5. Find x-and y-intercepts.
6. Recognize equations of horizontal and vertical lines and lines passing through the origin.
7. Use the midpoint formula.
8. Use a graphing calculator to graph an equation.
Objective 1

Interpret a line graph.
Personal Spending on Medical Care

Spending (in billions of dollars)

Year

Spending (in billions of dollars)


Source: U.S. Centers for Medicare and Medicaid Services.
Objective

2

Plot ordered pairs.
Each of the pair of numbers (3, 1), (−5, 6), and (4, −1) is an example of an ordered pair.

The position of any point in a plane is determined by referring to the horizontal number line, or $x$-axis, and the vertical number line, or $y$-axis.
The first number in the ordered pair indicates the position relative to the $x$-axis, and the second number indicates the position relative to the $y$-axis.

The $x$-axis and the $y$-axis make up a **rectangular** (or Cartesian), coordinate system.
Objective 3

Find ordered pairs that satisfy a given equation.
EXAMPLE 1

Complete the table of ordered pairs for $3x - 4y = 12$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>−3</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>−6</td>
<td>−2</td>
</tr>
</tbody>
</table>

a. (0, __)

Replace $x$ with 0 in the equation to find $y$.

$3x - 4y = 12$

$3(0) - 4y = 12$

$0 - 4y = 12$

$-4y = 12$

$y = -3$
continued

Complete the table of ordered pairs for $3x - 4y = 12$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$-3$</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
</tr>
<tr>
<td>$-6$</td>
<td>$-2$</td>
</tr>
</tbody>
</table>

b. $(__, 0)$

Replace $y$ with 0 in the equation to find $x$.

\[
3x - 4y = 12
\]

\[
3x - 4(0) = 12
\]

\[
3x - 0 = 12
\]

\[
3x = 12
\]

\[
x = 4
\]
Complete the table of ordered pairs for $3x - 4y = 12$. 

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-3</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
</tr>
<tr>
<td>4/3</td>
<td>-2</td>
</tr>
<tr>
<td>-6</td>
<td></td>
</tr>
</tbody>
</table>

c. (___, -2)

Replace $y$ with $-2$ in the equation to find $x$.

\[
3x - 4y = 12 \\
3x - 4(-2) = 12 \\
3x + 8 = 12 \\
3x = 4 \\
x = 4/3
\]
Complete the table of ordered pairs for $3x - 4y = 12$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-3</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
</tr>
<tr>
<td>4/3</td>
<td>-2</td>
</tr>
<tr>
<td>-6</td>
<td>-15/2</td>
</tr>
</tbody>
</table>

d. $(-6, ___)$

Replace $x$ with $-6$ in the equation to find $y$.

$$3x - 4y = 12$$

$$3(-6) - 4y = 12$$

$$-18 - 4y = 12$$

$$-4y = 30$$

$$y = -15/2$$
Objective

Graph lines.
The graph of an equation is the set of points corresponding to all ordered pairs that satisfy the equation. It gives a “picture” of the equation.

### Linear Equation in Two Variables

A linear equation in two variables can be written in the form

$$Ax + By = C,$$

where $A$, $B$, and $C$ are real numbers, $(A$ and $B$ not both 0). This form is called standard form.
Objective 5

Find $x$-and $y$-intercepts.
The *x-intercept* is the point (if any) where the line intersects the *x*-axis; likewise, the *y-intercept* is the point (if any) where the line intersects the *y*-axis.
Finding Intercepts

When graphing the equation of a line,

let \( y = 0 \) to find the \( x \)-intercept;

let \( x = 0 \) to find the \( y \)-intercept.
EXAMPLE 2

Find the $x$-and $y$-intercepts and graph the equation $2x - y = 4$.

$x$-intercept: Let $y = 0$.

$$2x - 0 = 4$$

$$2x = 4$$

$$x = 2 \quad (2, 0)$$

$y$-intercept: Let $x = 0$.

$$2(0) - y = 4$$

$$-y = 4$$

$$y = -4 \quad (0, -4)$$
Objective 6

Recognize equations of horizontal and vertical lines and lines passing through the origin.
EXAMPLE 3

Graph \( y = 3 \).

Writing \( y = 3 \) as \( 0x + 1y = 3 \) shows that any value of \( x \), including \( x = 0 \), gives \( y = 3 \). Since \( y \) is always 3, there is no value of \( x \) corresponding to \( y = 0 \), so the graph has no \( x \)-intercepts.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
</tr>
</tbody>
</table>

The graph \( y = 3 \) is a line not a point.
EXAMPLE 4

Graph \( x + 2 = 0 \).

\( 1x + 0y = -2 \)

shows that any value of \( y \), leads to \( x = -2 \), making the \( x \)-intercept \((-2, 0)\).

No value of \( y \) makes \( x = 0 \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>0</td>
</tr>
<tr>
<td>-2</td>
<td>2</td>
</tr>
</tbody>
</table>

The graph \( x + 2 = 0 \) is not just a point.

The graph is a line.
EXAMPLE 5

Graph $3x - y = 0$.

Find the intercepts.

$x$-intercept: Let $y = 0$.

$$3x - 0 = 0$$
$$3x = 0$$
$$x = 0$$

$y$-intercept: Let $x = 0$.

$$3(0) - y = 0$$
$$-y = 0$$
$$y = 0$$

The $x$-intercept is $(0, 0)$.

The $y$-intercept is $(0, 0)$. 
Objective 7

Use the midpoint formula.
Midpoint Formula

If the endpoints of a line segment $PQ$ are $(x_1, y_1)$ and $(x_2, y_2)$, its midpoint $M$ is

$$M = \left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right).$$
EXAMPLE 6

Find the coordinates of the midpoint of line segment \(PQ\) with endpoints \(P(-5, 8)\) and \(Q(2, 4)\).

Use the midpoint formula with \(x_1 = -5\), \(x_2 = 2\), \(y_1 = 8\), and \(y_2 = 4\):

\[
\left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right) = \left( \frac{-5 + 2}{2}, \frac{8 + 4}{2} \right) = \left( \frac{-3}{2}, \frac{12}{2} \right) = (-1.5, 6)
\]
Objective 8

Use a graphing calculator to graph an equation.
EXAMPLE 7

The graphing calculator screens shown below show the graph of a linear equation. What are the intercepts?

$x$-intercept $(-2, 0)$  
$y$-intercept $(0, 3)$
The Slope of a Line

1. Find the slope of a line, given two points on the line.
2. Find the slope of a line, given an equation of the line.
3. Graph a line, given its slope and a point on the line.
4. Use slopes to determine whether two lines are parallel, perpendicular, or neither.
5. Solve problems involving average rate of change.
Objective 1

Find the slope of a line, given two points on the line.
Slope Formula

The **slope** of the line through the distinct points \((x_1, y_1)\) and \((x_2, y_2)\) is

\[
m = \frac{\text{rise}}{\text{run}} = \frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} \quad x_1 \neq x_2.
\]
EXAMPLE 1

Find the slope of the line through points \((-6, 9)\) and \((3, -5)\).

If \((-6, 9) = (x_1, y_1)\) and \((3, -5) = (x_2, y_2)\), then

\[
m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-5 - 9}{3 - (-6)} = \frac{-14}{9} = -\frac{14}{9}
\]

Thus the slope is \(-\frac{14}{9}\).
If the ordered pairs are interchanged so that $(-6, 9) = (x_2, y_2)$, and $(3, -5) = (x_1, y_1)$ in the slope formula, the slope is the same.

$$m = \frac{9 - (-5)}{-6 - 3} = \frac{14}{-9} = -\frac{14}{9}$$

y-values are in the numerator, x-values are in the denominator.
Objective 2

Find the slope of a line, given an equation of the line.
EXAMPLE 2

Find the slope of the line $3x - 4y = 12$.

The intercepts can be used as two different points needed to find the slope. Let $y = 0$ to find that the $x$-intercept is $(4, 0)$. Then let $x = 0$ to find that the $y$-intercept is $(0, -3)$. Use the two points in the slope formula.

\[
m = \frac{\text{rise}}{\text{run}} = \frac{-3 - 0}{0 - 4} = \frac{-3}{-4} = \frac{3}{4}
\]

The slope is $\frac{3}{4}$.
EXAMPLE 3

Find the slope of each line.

a. \( y + 3 = 0 \)

To find the slope of the line with equation \( y + 3 = 0 \), select two different points on the line such as \((0, -3)\) and \((2, -3)\), and use the slope formula.

\[
m = \frac{-3 - (-3)}{2 - 0} = \frac{0}{2} = 0
\]

The slope is 0.
continued

b. \( x = -6 \)

To find the slope of the line with equation \( x = -6 \), select two different points on the line such as \((-6, 0)\) and \((-6, 3)\), and use the slope formula.

\[
m = \frac{3 - 0}{-6 - (-6)} = \frac{3}{0}
\]

The slope is undefined.
Horizontal and Vertical Lines

An equation of the form $y = b$ always intersects the $y$-axis; thus, the line with that equation is horizontal and has slope 0.

An equation of the form $x = a$ always intersects the $x$-axis; thus, the line with that equation is vertical and has undefined slope.
EXAMPLE 4

Find the slope of the graph of \( 3x + 4y = 9 \).

Solve the equation for \( y \).

\[
3x + 4y = 9
\]

\[
4y = -3x + 9
\]

\[
y = -\frac{3}{4}x + 9
\]

The slope is \(-\frac{3}{4}\).
Objective 3

Graph a line, given its slope and a point on the line.
EXAMPLE 5

Graph the line passing through \((-3, -2)\) that has slope \(\frac{1}{2}\).

Locate the point \((-3, -2)\) on the graph. Use the slope formula to find a second point on the line.

\[
m = \frac{\text{change in } y}{\text{change in } x} = \frac{1}{2},
\]

From \((-3, -2)\), move up 1 and then 2 units to the right to \((-1, -1)\).

Draw a line through the two points.
# Orientation of a Line in the Plane

A positive slope indicates that the line goes *up* (rises) from left to right.

A negative slope indicates that the line goes *down* (falls) from left to right.
Objective 4

Use slopes to determine whether two lines are parallel, perpendicular, or neither.
**Slopes of Parallel Lines**

Two nonvertical lines with the same slope are parallel.

Two nonvertical parallel lines have the same slope.
EXAMPLE 6

Determine whether the line through \((-1, 2)\) and \((3, 5)\) is parallel to the line through \((4, 7)\) and \((8, 10)\).

The line through \((-1, 2)\) and \((3, 5)\) has slope

\[
m_1 = \frac{5 - 2}{3 - (-1)} = \frac{3}{4}.
\]

The line through \((4, 7)\) and \((8, 10)\) has slope

\[
m_2 = \frac{10 - 7}{8 - 4} = \frac{3}{4}.
\]

The slopes are the same, so the lines are parallel.
Slopes of Perpendicular Lines

Two perpendicular lines, neither of which is parallel to an axis, have slopes that are negative reciprocals; that is, their product is \(-1\). Also, lines with slopes that are negative reciprocals are perpendicular.
EXAMPLE 7

Determine whether the lines with equations $3x + 5y = 6$ and $5x - 3y = 2$ are perpendicular.

Find the slope of each line by solving each equation for $y$.

$3x + 5y = 6$

$5y = -3x + 6$

$y = -\frac{3}{5}x + \frac{6}{5}$

$5x - 3y = 2$

$-3y = -5x + 2$

$y = \frac{5}{3}x - \frac{2}{3}$

Since the product of the slopes of the two lines is $\left(-\frac{3}{5}\right)\left(\frac{5}{3}\right) = -1$, the lines are perpendicular.
Objective 5

Solve problems involving average rate of change.
EXAMPLE 8

Use the ordered pairs of the data, \((2002, 818)\) and \((2003, 843)\), to find the average rate of change. How does it compare with the average rate of change found on this graph shown below?

\[(x_1, y_1) = (2002, 818)\]

\[(x_2, y_2) = (2002, 843)\]
Average rate of change = \frac{y_2 - y_1}{x_2 - x_1}

= \frac{843 - 818}{2003 - 2002}

= \frac{25}{1} = 25

The average rate of change is 25, which is approximately the same as Example 8.
EXAMPLE 9

In 1997, 36.4% of high school students smoked. In 2003, 21.9% of high school students smoked. Find the average rate of change in percent per year.

(Source: U.S. Centers for Disease Control and Prevention)

\[(x_1, y_1) = (1997, 36.4) \quad (x_2, y_2) = (2003, 21.9)\]

Average rate of change = \[
\frac{y_2 - y_1}{x_2 - x_1}
\]

\[
= \frac{21.9 - 36.4}{2003 - 1997} = \frac{-14.5}{6} \approx -2.417
\]

The average rate of change from 1997 to 2003 was \(-2.417\)% per year.
Linear Equations in Two Variables

1. Write an equation of a line, given its slope and \( y \)-intercept.
2. Graph a line, using its slope and \( y \)-intercept.
3. Write an equation of a line, given its slope and a point on the line.
4. Write an equation of a line, given two points on the line.
5. Write an equation of a line parallel or perpendicular to a given line.
6. Write an equation of a line that models real data.
7. Use a graphing calculator to solve linear equations in one variable.
Objective 1

Write an equation of a line, given its slope and y-intercept.
Slope-Intercept Form

\[ m = \frac{y - b}{x - 0} \]

\[ m = \frac{y - b}{x} \]

\[ mx = y - b \]

\[ mx + b = y \]

\[ y = mx + b \]
The **slope-intercept form** of the equation of a line with slope $m$ and $y$-intercept $(0, b)$ is

$$y = mx + b.$$
EXAMPLE 1

Find an equation of the line with slope 2 and y-intercept (0, –3).

\[ m = 2 \]
\[ b = -3 \]

Substitute these values into the slope-intercept form.

\[ y = mx + b \]
\[ y = 2x - 3 \]
Objective 2

Graph a line, using its slope and y-intercept.
EXAMPLE 2

Graph the line, using the slope and y-intercept.

\[ x + 2y = -4 \]

Write the equation in slope-intercept form by solving for \( y \).

\[ x + 2y = -4 \]
\[ 2y = -x - 4 \]
\[ y = \frac{-1}{2}x - 2 \]

Subtract \( x \).

Slope \( \frac{-1}{2} \) \( y \)-intercept (0, -2)
continued

Graph: \( y = -\frac{1}{2} x - 2 \)

1. Plot the \( y \)-intercept. 
   
   \( (0, -2) \)

2. The slope is \( \frac{-1}{2} \) or \( \frac{1}{-2} \).

3. Using \((-1/2)\), begin at \((0, -2)\) and move 1 unit \textit{down} and 2 units \textit{right}.

4. The line through these two points is the required graph.
Objective 3

Write an equation of a line, given its slope and a point on the line.
Point-Slope Form

The **point-slope form** of the equation of a line with slope \( m \) passing through the point \((x_1, y_1)\) is

\[
y - y_1 = m(x - x_1).
\]

Slope

Given point
EXAMPLE 3

Find an equation of the line with slope $\frac{2}{3}$ and passing through the point $(3, -4)$.

Use the point-slope form with $(x_1, y_1) = (3, -4)$ and $m = \frac{2}{3}$.

\[ y - y_1 = m(x - x_1) \]

\[ y - (-4) = \frac{2}{5}(x - 3) \]

Substitute

\[ y + 4 = \frac{2}{5}(x - 3) \]

Multiply by 5.

\[ 5y + 20 = 2x - 6 \]

Subtract 2x and 20.

\[ -2x + 5y = -26 \]

Multiply by $-1$.

\[ 2x - 5y = 26 \]
Objective 4

Write an equation of a line, given two points on the line.
EXAMPLE 4

Find an equation of the line passing through the points \((-2, 6)\) and \((1, 4)\). Write the equation in standard form.

First find the slope by the slope formula.

\[
m = \frac{4 - 6}{1 - (-2)} = \frac{-2}{3} = -\frac{2}{3}
\]

Use either point as \((x_1, y_1)\) in the point-slope form of the equation of a line.

Using the point \((1, 4)\): \(x_1 = 1\) and \(y_1 = 4\)
continued

\[ m = -\frac{2}{3}; \ x_1 = 1 \text{ and } y_1 = 4 \]

\[ y - y_1 = m(x - x_1) \]

\[ y - 4 = -\frac{2}{3}(x - 1) \quad \text{Substitute} \]

\[ 3y - 12 = -2x + 2 \quad \text{Multiply by 3.} \]

\[ 2x + 3y = 14 \quad \text{Add } 2x \text{ and } 12. \]

If the other point were used, the same equation would result.
Equations of Horizontal and Vertical Lines

The horizontal line through the point \((a, b)\) has equation \(y = b\).

The vertical line through the point \((a, b)\) has equation \(x = a\).
Objective 5

Write an equation of a line parallel or perpendicular to a given line.
EXAMPLE 5

Find an equation of the line passing through the point (–8, 3) and

a. parallel to the line $2x - 3y = 10$;
b. perpendicular to the line $2x - 3y = 10$.

Write each equation in slope-intercept form.

a. Find the slope of the line $2x - 3y = 10$ by solving for $y$.

$\begin{align*}
2x - 3y &= 10 \\
-3y &= -2x + 10 \\
y &= \frac{2}{3}x - \frac{10}{3}
\end{align*}$
The slope is \( \frac{2}{3} \).

Parallel lines have the same slope. Use point slope form and the given point.

\[
y - y_1 = m(x - x_1)
\]

\[
y - 3 = \frac{2}{3}[x - (-8)]
\]

\[
y - 3 = \frac{2}{3}(x + 8)
\]

\[
y - 3 = \frac{2}{3}x + \frac{16}{3}
\]

Find an equation of the line passing through the point \((-8, 3)\).

\[
y = \frac{2}{3}x + \frac{16}{3} + \frac{9}{3}
\]

\[
y = \frac{2}{3}x + \frac{25}{3}
\]

The fractions were not cleared because we want the equation in slope-intercept form instead of standard form.
Find an equation of the line passing through the point \((-8, 3)\).

b. Perpendicular lines. The slope is the negative reciprocal of \(\frac{2}{3}\).

Use point slope form and the given point. \(m = -\frac{3}{2}\)

\[
y - y_1 = m(x - x_1)
\]

\[
y - 3 = -\frac{3}{2}[x - (-8)]
\]

\[
y - 3 = -\frac{3}{2}(x + 8)
\]

\[
y - 3 = -\frac{3}{2}x - 12
\]

\[
y = -\frac{3}{2}x - 9
\]
### Forms of Linear Equations

<table>
<thead>
<tr>
<th>Equation</th>
<th>Description</th>
<th>When to Use</th>
</tr>
</thead>
</table>
| \( y = mx + b \) | **Slope-Intercept Form**  
Slope is \( m \).  
y-intercept is \((0, b)\). | The slope and \( y \)-intercept can be easily identified and used to quickly graph the equation. |
| \( y - y_1 = m(x - x_1) \) | **Point-Slope Form**  
Slope is \( m \).  
Line passes through \((x_1, y_1)\). | This form is ideal for finding the equation of a line if the slope and a point on the line or two points on the line are known. |
| \( Ax + By = C \) | **Standard Form**  
\((A, B, \text{ and } C \text{ integers, } A \geq 0)\)  
Slope is \(-\frac{A}{B} \text{ (} B \neq 0\).  
x-intercept is \((\frac{C}{A}, 0) \text{ (} A \neq 0\).  
y-intercept is \((0, \frac{C}{B}) \text{ (} B \neq 0\). | The \( x \)- and \( y \)-intercepts can be found quickly and used to graph the equation. The slope must be calculated. |
| \( y = b \) | **Horizontal Line**  
Slope is 0.  
y-intercept is \((0, b)\). | If the graph intersects only the \( y \)-axis, then \( y \) is the only variable in the equation. |
| \( x = a \) | **Vertical Line**  
Slope is undefined.  
x-intercept is \((a, 0)\). | If the graph intersects only the \( x \)-axis, then \( x \) is the only variable in the equation. |
Objective 6

Write an equation of a line that models real data.
EXAMPLE 6

Suppose there is a flat rate of $0.20 plus a charge of $0.10 per minute to make a telephone call. Write an equation that gives the cost $y$ for a call of $x$ minutes.

\[ y = 0.20 + 0.10x \]

or

\[ y = 0.10x + 0.20 \]
EXAMPLE 7

The percentage of the U.S. population 25 years and older with at least a high school diploma is shown in the table for selected years. Find an equation that models the data, using $x = 0$ to represent 1940, $x = 10$ to represent 1950, and so on.

<table>
<thead>
<tr>
<th>Year</th>
<th>Percent</th>
</tr>
</thead>
<tbody>
<tr>
<td>1940</td>
<td>24.5</td>
</tr>
<tr>
<td>1950</td>
<td>34.3</td>
</tr>
<tr>
<td>1960</td>
<td>41.4</td>
</tr>
<tr>
<td>1970</td>
<td>52.3</td>
</tr>
<tr>
<td>1980</td>
<td>66.5</td>
</tr>
<tr>
<td>1990</td>
<td>75.2</td>
</tr>
<tr>
<td>2000</td>
<td>80.4</td>
</tr>
</tbody>
</table>
Choose two data points and find the slope. Use 1940 and 2000.

\[ m = \frac{80.4 - 24.5}{2000 - 1940} = \frac{55.9}{60} = 0.9316 \]

The \( y \)-intercept is (0, 24.5).
The equation is:

\[ y = 0.93x + 24.5 \]

Selecting two different ordered pairs will lead to a different equation.
EXAMPLE 8

Use the ordered pairs (11, 164) and (13, 203) to find an equation that models the data in the graph below.

\[
m = \frac{203 - 164}{13 - 11} = \frac{39}{2} = 19.5
\]

Use the point-slope form with (11, 164).

\[
y - y_1 = m(x - x_1)
\]

\[
y - 164 = 19.5(x - 11)
\]

\[
y - 164 = 19.5x - 214.5
\]

\[
y = 19.5x - 50.5
\]
Objective

Use a graphing calculator to solve linear equations in one variable.
EXAMPLE 9

Use a graphing calculator to solve

\[ 4(x - 3) - x = x - 6 \]

Write the equation as an equivalent equation with 0 on one side.

\[ 4(x - 3) - x - (x - 6) = 0 \]

Then we graph

\[ Y_1 = 4(x - 3) - x - (x - 6) \]

The \( x \)-intercept is (3, 0).

The solution set is \( \{3\} \).
3.4 Linear Inequalities in Two Variables

1. Graph linear inequalities in two variables.
2. Graph the intersection of two linear inequalities.
3. Graph the union of two linear inequalities.
4. Use a graphing calculator to solve linear inequalities in one variable.
Objective 1

Graph linear inequalities in two variables.
Linear Inequalities in Two Variables

An inequality that can be written as

\[ Ax + By < C \quad \text{or} \quad Ax + By > C, \]

where \( A, B, \) and \( C \) are real numbers and \( A \) and \( B \) are not both 0, is a linear inequality in two variables.
The symbols $\leq$ and $\geq$ may replace $<$ and $>$ in the definition.

Consider the graph. The graph of the line $x + y = 5$ divides the points in the rectangular coordinate system into three sets:

1. Those points that lie on the line itself and satisfy the equation $x + y = 5$. [like (0, 5), (2, 3), and (5, 0)];
2. Those that lie in the half-plane above the line and satisfy the inequality $x + y > 5$ [like (5, 3) and (2, 4)];
3. Those that lie in the half-plane below the line and satisfy the inequality $x + y < 5$ [like (0, 0) and ($-3, -1$)].
Graphing a Linear Inequality

**Step 1** Draw the graph of the straight line that is the boundary. Make the line solid if the inequality involves ≤ or ≥. Make the line dashed if the inequality involves < and >.

**Step 2** Choose a test point. Choose any point not on the line, and substitute the coordinates of that point in the inequality.

**Step 3** Shade the appropriate region. Shade the region that includes the test point of it satisfies the original inequality. Otherwise, shade the region on the other side of the boundary line.
**CAUTION** When drawing the boundary line, be careful to draw a solid line if the inequality includes equality (≤, ≥) or a dashed line if equality is not included (<, >). Students often make errors in this step.
EXAMPLE 1

Graph $x + y \leq 4$.

**Step 1** Graph the line $x + y = 4$, which has intercepts (4,0) and (0,4), as a solid line since the inequality involves “≤”.

**Step 2** Test (0, 0).

\[
x + y \leq 4 \\
0 + 0 \leq 4 \\
0 \leq 4
\]

True
Step 3 Since the result is true, shade the region that contains (0, 0).
If the inequality is written in the form $y > mx + b$ or $y < mx + b$, then the inequality symbol indicates which half-plane to shade.

If $y > mx + b$, then shade above the boundary line;
If $y < mx + b$, then shade below the boundary line;

This method works only if the inequality is solved for $y$. 
EXAMPLE 2

Graph $3x + 4y < 12$.

Solve the inequality for $y$.

$$4y < -3x + 12$$
$$y < -\frac{3}{4}x + 3$$

Graph the boundary line $y = -\frac{3}{4}x + 3$ as a dashed line because the inequality symbol is $<$. Since the inequality is solved for $y$ and the inequality symbol is $<$, we shade the half-plane below the boundary line.
continued

As a check, choose a test point not on the line, say \((0, 0)\), and substitute for \(x\) and \(y\) in the original inequality.

\[
3x + 4y < 12
\]

\[
3(0) + 4(0) < 12
\]

\[
0 < 12 \quad \text{True}
\]

This decision agrees with the decision to shade below the line.
Objective 2

Graph the intersection of two inequalities.
EXAMPLE 3

Graph $x - y \leq 4$ and $x \geq -2$.

To begin graph each inequality separately.

$x - y \leq 4$

$x \geq -2$
Then we use shading to identify the intersection of the graphs.

\[ x - y \leq 4 \text{ and } x \geq -2 \]
Objective 3

Graph the union of two linear inequalities.
EXAMPLE 4

Graph $7x - 3y < 21$ or $x > 2$

Graph each inequality with dashed line.

The graph of the union is the region that includes all points on both graphs.

$7x - 3y < 21$ or $x > 2$
Objective 4

Use a graphing calculator to solve linear inequalities in one variable.
Introduction to Functions

1. Distinguish between independent and dependent variables.
2. Define and identify relations and functions.
3. Find the domain and range.
4. Identify functions defined by graphs and equations.
5. Use function notation.
6. Graph linear and constant functions.
Objective 1

Distinguish between independent and dependent variables.
We often describe one quantity in terms of another: The amount of your paycheck if you are paid hourly depends on the number of hours you worked.

The cost at the gas station depends on the number of gallons of gas you pumped into your car.

The distance traveled by a car moving at a constant speed depends on the time traveled.

If the value of the variable \( y \) depends on the value of the variable \( x \), then \( y \) is the dependent variable and \( x \) is the independent variable.
Objective

Define and identify relations and functions.
Relation

A relation is a set of ordered pairs.

Function

A function is a relation in which, for each value of the first component of the ordered pairs, there is exactly one value of the second component.
EXAMPLE 1

Determine whether each relation defines a function.

a. \{(0, 3), (–1, 2), (–1, 3)\}

No, the same \(x\)-value is paired with a different \(y\)-value.

*In a function, no two ordered pairs can have the same first component and different second components.*

b. \{(5, 4), (6, 4), (7, 4)\}

Yes, each different \(x\)-value is paired with a \(y\)-value. This does not violate the definition of a function.
Relations and functions can also be expressed as a correspondence or *mapping* from one set to another.

Function

Not a function
Objective 3

Find the domain and range.
Domain and Range

In a relation, the set of all values of the independent variable $(x)$ is the **domain**.

The set of all values of the dependent variable $(y)$ is the **range**.
EXAMPLE 2

Give the domain and range of the relation represented by the table for cellular telephone subscribers. Does it define a function?

<table>
<thead>
<tr>
<th>Year</th>
<th>Subscribers</th>
</tr>
</thead>
<tbody>
<tr>
<td>1999</td>
<td>86,047</td>
</tr>
<tr>
<td>2000</td>
<td>109,478</td>
</tr>
<tr>
<td>2001</td>
<td>128,375</td>
</tr>
<tr>
<td>2002</td>
<td>140,767</td>
</tr>
<tr>
<td>2003</td>
<td>158,722</td>
</tr>
<tr>
<td>2004</td>
<td>182,140</td>
</tr>
</tbody>
</table>


Range: \{86,047, 109,478, 128,375, 140,767, 158,722, 182,140\}

Yes; it is a function.
EXAMPLE 3

Give the domain and range of the relation.

The arrowheads indicate that the line extends indefinitely left and right.

**Domain:** $(\infty, \infty)$

**Range:** $(-\infty, 4]$
## Agreement on Domain

Unless specified otherwise, the domain of a relation is assumed to be all real numbers that produce real numbers when substituted for the independent variable.
Objective 4

Identify functions defined by graphs and equations.
Vertical Line Test

If every vertical line intersects the graph of a relation in no more than one point, then the relation is a function.
Function

Not a Function
EXAMPLE 4

Use the vertical line test to decide whether the relation shown below is a function.

Yes, the relation is a function.
EXAMPLE 5

Decide whether each equation defines $y$ as a function of $x$, and give the domain.

a. $y = -2x + 7$

$y$ is always found by multiplying by negative two and adding 7. Each value of $x$ corresponds to just one value of $y$.  

Yes, $(-\infty, \infty)$

b. $y = \sqrt{5x - 6}$  

$5x - 6 \geq 0$

$5x \geq 6$

$x \geq \frac{6}{5}$  

Yes, $\left[ \frac{6}{5}, \infty \right)$
c. \( y^4 = x \) \hspace{1cm} \text{No, } [0, \infty)

d. \( y \geq 4x + 2 \) \hspace{1cm} 4x + 2 \geq 0
\hspace{1cm} 4x \geq -2
\hspace{1cm} x \geq -\frac{1}{2}
\hspace{1cm} \text{No, } (-\infty, \infty)

e. \( y = \frac{6}{5+3x} \) \hspace{1cm} 0 = 5 + 3x
\hspace{1cm} -5 = 3x
\hspace{1cm} \frac{5}{3} = x
\hspace{1cm} \text{Yes} \hspace{1cm} (-\infty, -\frac{5}{3}) \cup (-\frac{5}{3}, \infty)

The denominator would be zero and this is undefined so it is not included in the domain.
Variations of the Definition of a Function

1. A function is a relation in which, for each value of the first component of the ordered pairs, there is exactly one value of the second component.

2. A function is a set of distinct ordered pairs in which no first component is repeated.

3. A function is a rule or correspondence that assigns exactly one range value to each domain value.
Objective 5

Use function notation.
When a function $f$ is defined with a rule or an equation using $x$ and $y$ for the independent and dependent variables, we say, “$y$ is a function of $x$” to emphasize that $y$ depends on $x$. We use the notation

$$y = f(x),$$

called function notation, to express this and read $f(x)$ as “$f$ of $x$.”
EXAMPLE 6

Let \( f(x) = \frac{-3x + 5}{2} \). Find

a. \( f(-3) \)

\[
f(-3) = \frac{-3(-3) + 5}{2} = \frac{9 + 5}{2} = \frac{14}{2} = 7
\]

b. \( f(t) \)

\[
f(t) = \frac{-3(t) + 5}{2}
\]
EXAMPLE 7

Let \( g(x) = 5x - 1 \). Find and simplify \( g(m + 2) \).

\[
\begin{align*}
g(x) &= 5x - 1 \\
g(m + 2) &= 5(m + 2) - 1 \\
&= 5m + 10 - 1 \\
&= 5m + 9
\end{align*}
\]
EXAMPLE 8

Find \( f(2) \) for each function.

a. \[
\begin{array}{c|c}
 x & f(x) \\
-4 & 16 \\
-2 & 4 \\
0 & 0 \\
2 & 4 \\
4 & 16 \\
\end{array}
\]

\[ f(2) = 4 \]

b. \( f = \{(2, 6), (4, 2)\} \)

\[ f(2) = 6 \]
c. \( f(x) = -x^2 \)
\[ f(2) = -2^2 \]
\[ f(2) = -4 \]

d. The function graphed.

\[ f(2) = 3 \]
Finding an Expression for $f(x)$

<table>
<thead>
<tr>
<th>Step 1</th>
<th>Solve the equation for $y$.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Step 2</td>
<td>Replace $y$ with $f(x)$.</td>
</tr>
</tbody>
</table>
EXAMPLE 9

Rewrite the equation using function notation \( f(x) \). Then find \( f(1) \) and \( f(a) \).

\[
x^2 - 4y = 3
\]

Step 1: Solve for \( y \).

\[
-4y = -x^2 + 3
\]

\[
y = \frac{-x^2}{-4} + \frac{3}{-4}
\]

\[
y = \frac{x^2}{4} - \frac{3}{4}
\]

\[
f(x) = \frac{x^2}{4} - \frac{3}{4}
\]
continued

\[ f(1) = \frac{x^2}{4} - \frac{3}{4} \]

\[ f(1) = \frac{(1)^2}{4} - \frac{3}{4} = \frac{1}{4} - \frac{3}{4} = -\frac{1}{2} \]

\[ f(a) = \frac{x^2}{4} - \frac{3}{4} \]

\[ f(a) = \frac{(a)^2}{4} - \frac{3}{4} = \frac{a^2 - 3}{4} \]
Objective 6

Graph linear and constant functions.
A function that can be defined by

\[ f(x) = ax + b \]

for real numbers \( a \) and \( b \) is a \textbf{linear function}. The value of \( a \) is the slope \( m \) of the graph of the function.
EXAMPLE 10

Graph \( f(x) = \frac{1}{2} x - \frac{3}{2} \)

\( f(x) = \frac{1}{2} x - \frac{3}{2} \)

Slope \( y \)-intercept