Systems of Linear Equations in Two Variables

1. Decide whether an ordered pair is a solution of a linear system.
2. Solve linear systems by graphing.
3. Solve linear systems (with two equations and two variables) by substitution.
4. Solve linear systems (with two equations and two variables) by elimination.
5. Solve special systems.
6. Recognize how a graphing calculator is used to solve a linear system.
A set of equations is called a **system of equations**.

The **solution set of a linear system** of equations contains all ordered pairs that satisfy all the equations of the system *at the same time*.

Linear system of equations

\[
\begin{align*}
2.5x + y &= 19.4 \\
-1.7x + y &= 4.4
\end{align*}
\]

(Here, \(x = 0\) represents 2000, \(x = 1\) represents 2001, and so on; \(y\) represents sales in millions of units.)

Source: Consumer Electronics Association.
Objective 1

Decide whether an ordered pair is a solution of a linear system.
EXAMPLE 1

Is the ordered pair a solution of the given system?

a. \((-4, 2)\) \quad 2x + y = -6

b. \((3, -12)\) \quad x + 3y = 2

Replace \(x\) with \(-4\) and \(y\) with 2 in each equation of the system.

\[
\begin{align*}
2x + y &= -6 \\
2(-4) + 2 &= -6 \\
-8 + 2 &= -6 \\
-6 &= -6 \quad \text{True}
\end{align*}
\]

\[
\begin{align*}
x + 3y &= 2 \\
-4 + 3(2) &= 2 \\
-4 + 6 &= 2 \\
2 &= 2 \quad \text{True}
\end{align*}
\]

Since \((-4, 2)\) makes both equations true, it is a solution.
b. (3, –12)

Replace \( x \) with 3 and \( y \) with –12.

\[
\begin{align*}
2x + y &= -6 \\
2(3) + (-12) &= -6 \\
6 - 12 &= -6 \\
-6 &= -6 \quad \text{True}
\end{align*}
\]

\[
\begin{align*}
x + 3y &= 2 \\
3 + 3(-12) &= 2 \\
3 - 36 &= 2 \\
-33 &= 2 \quad \text{False}
\end{align*}
\]

The ordered pair (3, –12) is not a solution of the system, since it does not make both equations true.
Objective 2

Solve linear systems by graphing.
EXAMPLE 2

Solve the system of equations by graphing.

Graph each linear equation.

\[2x + y = -5\]
\[-x + 3y = 6\]

\[y = -2x - 5\]
\[y = \frac{1}{3}x + 2\]

The graph suggests that the point of intersection is the ordered pair \((-3, 1)\).

Check the solution in both equations.
Graphs of Linear Systems in Two Variables

1. The two graphs intersect in a single point. The coordinates of this point give the only solution of the system. Since the system has a solution, it is consistent. The equations are not equivalent, so they are independent. See Figure 3(a).

2. The graphs are parallel lines. There is no solution common to both equations, so the solution set is \( \emptyset \) and the system is inconsistent. Since the equations are not equivalent, they are independent. See Figure 3(b).

3. The graphs are the same line. Since any solution of one equation of the system is a solution of the other, the solution set is an infinite set of ordered pairs representing the points on the line. This type of system is consistent because there is a solution. The equations are equivalent, so they are dependent. See Figure 3(c).

![Figure 3](image-url)
Objective 3

Solve linear systems (with two equations and two variables) by substitution.
It can be difficult to read exact coordinates, especially if they are not integers, from a graph. For this reason, we usually use algebraic methods to solve systems.

The **substitution method**, is most useful for solving linear systems in which one equation is solved or can easily be solved for one variable in terms of the other.
EXAMPLE 3

Solve the system.  

\[ 5x - 3y = -6 \quad (1) \]
\[ x = 2 - y \quad (2) \]

Since equation (2) is solved for \( x \), substitute \( 2 - y \) for \( x \) in equation (1).

\[ 5x - 3y = -6 \quad (1) \]
\[ 5(2 - y) - 3y = -6 \]
\[ 10 - 5y - 3y = -6 \]
\[ 10 - 8y = -6 \]
\[ -8y = -16 \]
\[ y = 2 \]

Be sure to use parentheses here.

Distributive property

Combine like terms.

Subtract 10.

Divide by –8.
We found $y$. Now find $x$ by substituting $2$ for $y$ in equation (2).

$$x = 2 - y$$

$$= 2 - 2 = 0$$

Thus $x = 0$ and $y = 2$, giving the ordered pair $(0, 2)$. Check this solution in both equations of the original system.

**Check:**

$$5x - 3y = -6$$  \hspace{1cm} (1)

$$x = 2 - y$$  \hspace{1cm} (2)

$$5(0) - 3(2) = -6$$

$$0 - 6 = -6$$

$$-6 = -6 \quad \text{True}$$

$$0 = 2 - 2$$

$$0 = 0 \quad \text{True}$$

The solution set is $(0, 2)$. 

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Solving a Linear System by Substitution

**Step 1** Solve one of the equations for either variable. If one of the equations has a variable term with coefficient 1 or \(-1\), choose that equation since the substitution method is usually easier this way.

**Step 2** Substitute for that variable in the other equation. The result should be an equation with just one variable.

**Step 3** Solve the equation from Step 2.

**Step 4** Find the other value. Substitute the result from Step 3 into the equation from Step 1 to find the value of the other variable.

**Step 5** Check the solution in both of the original equations. Then write the solution set.
EXAMPLE 4

Solve the system.  

\[ 4x + y = 5 \quad (1) \]
\[ 2x - 3y = 13 \quad (2) \]

**Step 1** Solve one of the equations for either \( x \) or \( y \).

\[ 4x + y = 5 \quad (1) \]

\[ y = 5 - 4x \quad \text{Subtract 4x.} \]

**Step 2** Substitute \( 5 - 4x \) into equation (2).

\[ 2x - 3(5 - 4x) = 13 \]

**Step 3** Solve.

\[ 2x - 15 + 12x = 13 \]
\[ 14x = 28 \]
\[ x = 2 \]
continued

**Step 4** Now find \( y \).

\[
y = 5 - 4x
\]

\[
y = 5 - 4(2) = -3
\]

**Step 5** Check the solution \((2, -3)\) in both equations.

\[
4x + y = 5 \quad (1)
\]

\[
2x - 3y = 13 \quad (2)
\]

\[
4(2) + (-3) = 5
\]

\[
8 - 3 = 5
\]

\[
5 = 5 \quad \text{True}
\]

\[
2(2) - 3(-3) = 13
\]

\[
4 + 9 = 13
\]

\[
13 = 13 \quad \text{True}
\]

The solution set is \((2, -3)\).
EXAMPLE 5

\[ -2x + 5y = 22 \quad (1) \]

Solve the system.
\[ \frac{1}{2} x + \frac{1}{4} y = \frac{1}{2} \quad (2) \]

Clear the fractions in equation (2). Multiply by the LCD, 4.
\[ 4 \left( \frac{1}{2} x + \frac{1}{4} y \right) = 4 \left( \frac{1}{2} \right) \]
\[ 4 \cdot \frac{1}{2} x + 4 \cdot \frac{1}{4} y = 4 \cdot \frac{1}{2} \]
\[ 2x + y = 2 \quad (3) \]

Solve equation (3) for \( y \).
\[ 2x + y = 2 \quad (3) \]
\[ y = 2 - 2x \]
continued

Substitute $y = 2 - 2x$ for $y$ in equation (1).

$$-2x + 5y = 22 \quad (1)$$

$$\frac{1}{2}x + \frac{1}{4}y = \frac{1}{2} \quad (2)$$

Substitute $y = 2 - 2x$ for $y$ in equation (1).

$$-2x + 5(2 - 2x) = 22$$

$$-2x + 10 - 10x = 22$$

$$-12x + 10 = 22$$

$$-12x = 12$$

$$x = -1$$

Solve $y$.

$$y = 2 - 2x$$

$$y = 2 - 2(-1)$$

$$y = 2 + 2 = 4$$

A check verifies that the solution set is $\{(-1, 4)\}$. 
Objective 4

Solve linear systems (with two equations and two variables) by elimination.
EXAMPLE 5

Solve the system.

\[-2x + 3y = -10 \quad (1)\]
\[2x + 2y = 5 \quad (2)\]

Adding the equations together will eliminate \(x\).

\[-2x + 3y = -10 \quad (1)\]
\[2x + 2y = 5 \quad (2)\]

\[
\begin{align*}
5y &= -5 \\
y &= -1
\end{align*}
\]

To find \(x\), substitute \(-1\) for \(y\) in either equation.

\[2x + 2y = 5 \quad (2)\]
\[2x + 2(-1) = 5\]
\[2x - 2 = 5\]
\[2x = 7\]
\[x = \frac{7}{2}\]

The solution set is \(\{(7/2, -1)\}\).
Solving a Linear System by Elimination

Step 1  Write both equations in standard form
        \[ Ax + By = C. \]

Step 2  Make the coefficients of one pair of variable terms opposites. Multiply one or both equations by appropriate numbers so that the sum of the coefficients of either the \( x \)- or \( y \)-terms is 0.

Step 3  Add the new equations to eliminate a variable. The sum should be an equation with just one variable.

Step 4  Solve the equation from Step 3 for the remaining variable.

Step 5  Find the other value. Substitute the result of Step 4 into either of the original equations and solve for the other variable.

Step 6  Check the solution in both of the original equations. Then write the solution set.
EXAMPLE 7

Solve the system.

\[2x + 3y = 19 \quad (1)\]
\[3x - 7y = -6 \quad (2)\]

**Step 1** Both equations are in standard form.

**Step 2** Select a variable to eliminate, say \(y\). Multiply equation (1) by 7 and equation (2) by 3.

\[14x + 21y = 133\]

**Step 3** Add.

\[9x - 21y = -18\]

\[23x = 115\]

**Step 4** Solve for \(x\).

\[x = 5\]
Step 5  To find \( y \) substitute 5 for \( x \) in either equation (1) or equation (2).

\[
2x + 3y = 19 \\
2(5) + 3y = 19 \\
10 + 3y = 19 \\
3y = 9 \\
y = 3
\]

Step 6  The solution is \((5, 3)\). To check substitute 5 for \( x \) and 3 for \( y \) in both equations (1) and (2).

The ordered pair checks, the solution set is \( \{(5, 3)\} \).
Objective 5

Solve special systems.
EXAMPLE 8

Solve the system.  
\[ 2x + y = 6 \quad (1) \]
\[ -8x - 4y = -24 \quad (2) \]

Multiply equation (1) by 4 and add the result to equation (2).  
\[ 8x + 4y = 24 \quad (1) \]
\[ -8x - 4y = -24 \quad (2) \]

\[ 0 = 0 \quad \text{True} \]

Equations (1) and (2) are equivalent and have the same graph. The equations are dependent.

The solution set is the set of all points on the line with equation \( 2x + y = 6 \), written in set-builder notation \( \{(x, y) | 2x + y = 6\} \).
EXAMPLE 9

Solve the system. \[\begin{align*}
4x - 3y &= 8 \quad (1) \\
8x - 6y &= 14 \quad (2)
\end{align*}\]

Multiply equation (1) by \(-2\) and add the result to equation (2). \[-8x + 6y = -16 \quad (1) \]
\[\begin{align*}
8x - 6y &= 14 \quad (2) \\
\hline
0 &= -2
\end{align*}\]

The result of adding the equations is a false statement, which indicates the system is inconsistent. The graphs would be parallel lines. There are no ordered pairs that satisfy both equations. The solution set is \(\emptyset\).
<table>
<thead>
<tr>
<th>Special Cases of Linear Systems</th>
</tr>
</thead>
<tbody>
<tr>
<td>If both variables are eliminated when a system of linear equations is solved,</td>
</tr>
<tr>
<td>1. there are infinitely many solutions if the resulting statement is <em>true</em>;</td>
</tr>
<tr>
<td>2. there is no solution if the resulting statement is <em>false</em>.</td>
</tr>
</tbody>
</table>
Objective 6

Recognize how a graphing calculator is used to solve a linear system.
EXAMPLE 10

Use a graphing calculator to solve the system.

\[2x + y = -5\]

\[-x + 3y = 6\]

Solve each equation for \(y\).

\[y = -5 - 2x\]

\[y = 2 + \frac{1}{3}x\]

The points of intersection are displayed at the bottom of the screen, indicating that the solution set is \((-3, 1)\).
4.2 Systems of Linear Equations in Three Variables

1. Understand the geometry of systems of equations in three variables.

2. Solve linear systems (with three equations and three variables) by elimination.

3. Solve linear systems (with three equations and three variables) in which some of the equations have missing terms.

4. Solve special systems.
A solution of an equation in three variables, such as $2x + 3y - z = 4$ is called an **ordered triple** and is written $(x, y, z)$. 
Objective 1

Understand the geometry of systems of equations in three variables.
The graph of a linear equation with three variables is a *plane*, not a line.

A number of possible solutions are shown below.

**FIGURE 7**

- **(a)** A single solution
- **(b)** Points of a line in common
- **(c)** All points in common
- **(d)** No points in common
- **(e)** No points in common
- **(f)** No points in common
- **(g)** No points in common
Graphs of Linear Systems in Three Variables

1. The three planes may meet at a single, common point that is the solution of the system. (See Figure 7a).

2. The three planes may have the points of a line in common, so that the infinite set of points that satisfy the equation of the line is the solution of the system. (See Figure 7b).

3. The three planes may coincide, so that the solution of the system is the set of all points on a plane. (See Figure 7c).

4. The planes may have no points common to all three, so that there is no solution of the system. (See Figures 7d-g.)
Objective 2

Solve linear systems (with three equations and three variables) by elimination.
Solving a Linear System in Three Variables

**Step 1** Eliminate a variable. Use the elimination method to eliminate any variable from any two of the original equations. The result is an equation in two variables.

**Step 2** Eliminate the same variable again. Eliminate the same variable from any other two equations. The result is an equation in the same two variables as in Step 1.

**Step 3** Eliminate a different variable and solve. Use the elimination method to eliminate a second variable from the two equations in two variables that result from Steps 1 and 2. The result is an equation in one variable which gives the value of that variable.
Solving a Linear System in Three Variables

**Step 4  Find a second value.** Substitute the value of the variables found in Step 3 into either of the equations in two variable to find the value of the second variable.

**Step 5  Find a third value.** Use the values of the two variables from Steps 3 and 4 to find the value of the third variable by substituting into an appropriate equation.

**Step 6  Check** the solution in all of the original equations. Then write the solution set.
EXAMPLE 1

Solve the system.

\[
\begin{align*}
\text{(1)} & \quad x + y + z = 2 \\
\text{(2)} & \quad x - y + 2z = 2 \\
\text{(3)} & \quad -x + 2y - z = 1
\end{align*}
\]

**Step 1** Eliminate \( y \) by adding equations (1) and (2).

\[
\begin{align*}
(1) & \quad x + y + z = 2 \\
(2) & \quad x - y + 2z = 2 \\
\hline
(4) & \quad 2x + 3z = 4
\end{align*}
\]

**Step 2** To eliminate \( y \) again, multiply equation (2) by 2 and add the result to equation (3).

\[
\begin{align*}
(2 \times 2) & \quad 2x - y + 2z = 2 \\
(3) & \quad -x + 2y - z = 1
\end{align*}
\]
continued

\[
\begin{align*}
2x - 2y + 4z &= 4 \quad (2 \times 2) \\
-x + 2y - z &= 1 \quad (3) \\
x + 3z &= 5 \quad (5)
\end{align*}
\]

**Step 3** Use equations (4) and (5) to eliminate \( z \). Multiply equation (5) by \(-1\) and add the result to equation (4).

\[
\begin{align*}
2x + 3z &= 4 \quad (4) \\
-x - 3z &= -5 \quad (5) \times -1 \\
x &= -1
\end{align*}
\]
Make sure equation (5) has the same variables as equation 4.
continued

**Step 4** Substitute \(-1\) for \(x\) in equation (5) to find \(z\).

\[
x + 3z = 5 \quad (5)
\]

\[
-1 + 3z = 5
\]

\[
3z = 6
\]

\[
z = 2
\]

**Step 5** Substitute \(-1\) for \(x\) and \(2\) for \(z\) in equation (1) to find \(y\).

\[
x + y + z = 2 \quad (1)
\]

\[
-1 + y + 2 = 2
\]

\[
y + 1 = 2
\]

\[
y = 1
\]
Step 6  Check. \((-1, 1, 2)\)

\[
\begin{align*}
  x + y + z &= 2 & (1) & -1 + 1 + 2 = 2 \\
  x - y + 2z &= 2 & (2) & -1 - 1 + 2(2) = 2 \\
  -x + 2y - z &= 1 & (3) & -( -1 ) + 2(1) - (2) = 1
\end{align*}
\]

The solution set is \{\((-1, 1, 2)\)\}. 

Write the values of \(x\), \(y\), and \(z\) in the correct order.
Objective 3

Solve linear systems (with three equations and three variables) in which some of the equations have missing terms.
EXAMPLE 2

Solve the system.

\[ x - y = 6 \quad (1) \]
\[ 2y + 5z = 1 \quad (2) \]
\[ 3x - 4z = 8 \quad (3) \]

Since equation (3) is missing \( y \), eliminate \( y \) again from equations (1) and (2). Multiply equation (1) by 2 and add the result to equation (2).

\[
\begin{align*}
2x - 2y &= 12 \quad (1) \times 2 \\
2y + 5z &= 1 \quad (2) \\
\hline
2x + 5z &= 13 \quad (4)
\end{align*}
\]
Use equation (4) together with equation (3) to eliminate \( x \). Multiply equation (4) by 3 and equation (3) by \(-2\). Then add the results.

\[
\begin{align*}
2x + 5z &= 13 \\
3x - 4z &= 8
\end{align*}
\]

\[
\begin{align*}
6x + 15z &= 39 \\
-6x + 8z &= -16
\end{align*}
\]

\[
23z = 23
\]

\[
z = 1
\]
continued

Substitute 1 for \( z \) in equation (2) to find \( y \).

\[ 2y + 5z = 1 \quad (2) \]
\[ 2y + 5(1) = 1 \]
\[ 2y + 5 = 1 \]
\[ 2y = -4 \]
\[ y = -2 \]

Substitute \(-2\) for \( y \) in (1) to find \( x \).

\[ x - y = 6 \quad (1) \]
\[ x - (-2) = 6 \]
\[ x + 2 = 6 \]
\[ x = 4 \]

Check \((4, -2, 1)\) in each of the original equations to verify that it is the solution set.
Objective 4

Solve special systems.
EXAMPLE 3

Solve the system.

\[ 3x - 5y + 2z = 1 \] \hspace{1cm} (1)
\[ 5x + 8y - z = 4 \] \hspace{1cm} (2)
\[ -6x + 10y - 4z = 5 \] \hspace{1cm} (3)

Multiply equation (1) by 2 and add the result to equation (3).

\[ 6x - 10y + 4z = 2 \] \hspace{1cm} (1) \times 2
\[ -6x + 10y - 4z = 5 \] \hspace{1cm} (3)

\[ 0 = 7 \]

Since a false statement results, the system is inconsistent. The solution set is \( \emptyset \).
EXAMPLE 4

Solve the system.

\[ x - y + z = 4 \]
\[ -3x + 3y - 3z = -12 \]
\[ 2x - 2y + 2z = 8 \]

Since equation (2) is \(-3\) times equation (1) and equation (3) is \(2\) times equation (1), the three equations are dependent. All three have the same graph.

The solution set is \( \{(x, y, z) \mid x - y + z = 4\} \).
EXAMPLE 5

Solve the system.

\[
\begin{align*}
2x + 3y - z &= 8 \\
\frac{1}{2}x + \frac{3}{4}y - \frac{1}{4}z &= 2 \\
x + \frac{3}{2}y - \frac{1}{2}z &= -6
\end{align*}
\]

Eliminate the fractions in equations (2) and (3).

Multiply equation (2) by 4.

\[
2x + 3y - z = 8 \quad (1)
\]

Multiply equation (3) by 2.

\[
2x + 3y - z = -12 \quad (5)
\]
EXAMPLE 5

Equations (1) and (4) are dependent (they have the same graph).

Equations (1) and (5) are not equivalent. Since they have the same coefficients but different constant terms, their graphs have no points in common (the planes are parallel).

Thus the system is inconsistent and the solution set is $\emptyset$.  

\[
\begin{align*}
2x + 3y - z &= 8 \\
2x + 3y - z &= 8 \\
2x + 3y - z &= -12
\end{align*}
\]
Applications of Systems of Linear Equations

1. Solve geometry problems by using two variables.
2. Solve money problems by using two variables.
3. Solve mixture problems by using two variables.
4. Solve distance–rate–time problems by using two variables.
5. Solve problems with three variables by using a system of three equations.
Solving an Applied Problem by Writing a System of Equations

Step 1  **Read** the problem carefully until you understand what is given and what is to be found.

Step 2  **Assign variables** to represent the unknown values, using diagrams or tables as needed. **Write down** what each variable represents.

Step 3  **Solve** the system of equations.

Step 4  **State the answer** to the problem. Does it seem reasonable?

Step 5  **Check** the answer in the words of the original problem.
Objective 1

Solve geometry problems by using two variables.
EXAMPLE 1

A rectangular soccer field has perimeter 360 yd. Its length is 20 yd more than its width. What are its dimensions?

Step 1 Read the problem again. We are asked to find the dimensions of the field.

Step 2 Assign variables.

Let $L =$ the length and $W =$ the width.

Step 3 Write a system of equations.

The perimeter of a rectangle is given by $2W + 2L = 360$.

Since the length is 20 yd more than the width, $L = W + 20$. 
The system is

\[ L = W + 20 \]  \hspace{0.5cm} (1)
\[ 2W + 2L = 360 \]  \hspace{0.5cm} (2)

**Step 4 Solve.** Substitute \( W + 20 \) for \( L \) in equations (2).

\[ 2W + 2(W + 20) = 360 \]
\[ 2W + 2W + 40 = 360 \]
\[ 4W = 320 \]
\[ W = 80 \]

Substitute \( W = 80 \) into equation (1).
\[ L = 80 + 20 = 100 \]
Step 5  State the answer.

The length of the field is 100 yards and the width is 80 yards.

Step 6  Check.

The perimeter of the soccer field is

$$2(100) + 2(80) = 360 \text{ yd},$$

and the length, 100 yards is 20 more than the width, since $100 - 20 = 80$.

The answer is correct.
Objective 2

Solve money problems by using two variables.
EXAMPLE 2

In recent Major League Baseball and National Football League seasons, based on average ticket prices, three baseball tickets and two football tickets would have cost $159.50, while two baseball tickets and one football ticket would have cost $89.66. What were the average ticket prices for the tickets for the two sports?
(Source: Team Marketing Report, Chicago.)

Step 1 Read the problem again. There are two unknowns.
continued

Step 2 Assign variables.

Let \( x \) = the average cost of baseball tickets, and \( y \) = the average cost of football tickets.

Step 3 Write a system of equations.

From the given information,

\[
3x + 2y = 159.50 \quad (1)
\]

\[
2x + y = 89.66. \quad (2)
\]
Step 4 Solve.

Multiply equation (2) by $-2$ and add to equation (1).

\[
\begin{align*}
3x + 2y &= 159.50 \\
-4x - 2y &= -179.32 \\
\hline
-x &= -19.82
\end{align*}
\]

\[x = 19.82\]

Let $x = 19.82$ in equation (2).

\[
\begin{align*}
2(19.82) + y &= 89.66 \\
39.64 + y &= 89.66 \\
y &= 50.02
\end{align*}
\]
continued

**Step 5  State the answer.**

The average cost of a baseball ticket is $19.82 and the average cost of a football ticket is $50.02.

**Step 6  Check.**

\[ 3(19.82) + 2(50.02) = 159.50 \]

and \[ 2(19.82) + 50.02 = 89.66. \]

The answer is correct.
Objective 3

Solve mixture problems by using two variables.
EXAMPLE 3

A grocer has some $4-per-lb coffee as some $8-per-lb coffee that she will mix to make 50 lb of $5.60-per-lb coffee. How many pounds of each should be used?

*Step 1* Read the problem.

*Step 2* Assign variables.

Let \( x \) = number of pounds of the $4-per-lb coffee and \( y \) = the number of pounds of the $8-per-pound coffee.
continued

<table>
<thead>
<tr>
<th>Price per Pound</th>
<th>Number of Pounds</th>
<th>Value of Coffee</th>
</tr>
</thead>
<tbody>
<tr>
<td>$4</td>
<td>$4x</td>
<td>4x</td>
</tr>
<tr>
<td>$8</td>
<td>$8y</td>
<td>8y</td>
</tr>
<tr>
<td>$5.60</td>
<td>50</td>
<td>5.6(50) = 280</td>
</tr>
</tbody>
</table>

**Step 3  Write a system of equations.**

From the columns;

\[
x + y = 50 \quad (1)
\]
\[
4x + 8y = 280 \quad (2)
\]
continued

**Step 4  Solve.**

To eliminate $x$, multiply equation (1) by $-4$ and add to equation (2).

$$-4x - 4y = -200 \quad -4 \quad (1)$$

$$4x + 8y = 280 \quad (2)$$

$$4y = 80$$

$$y = 20$$

Since $y = 20$ and $x + y = 50$, $x = 30$. 
Step 5  State the answer.

To mix the coffee, 30 lb of $4\text{-per-lb}$ coffee and 20 lb of $8\text{-per-lb}$ coffee should be used.

Step 6  Check.

\[30 + 20 = 50\]
and \[4(30) + 8(20) = 280.\]

The answer is correct.
Objective 4

Solve distance–rate–time problems by using two variables.
EXAMPLE 4

A train travels 600 mi in the same time that a truck travels 520 mi. Find the speed of each vehicle if the train’s average speed is 8 mph faster than the truck’s.

Step 1  Read the problem. We need to find the speed of each vehicle.

Step 2  Assign variables.

Let \( x \) = the train’s speed and
\[ y = \text{the truck’s speed}. \]
Step 3  Write a system of equations.

\[
\frac{600}{x} = \frac{520}{y}
\]

\[
600y = 520x
\]

\[-520x + 600y = 0 \quad (1)\]

\[x = y + 8. \quad (2)\]
continued

Step 4 Solve.

Substitute \( y + 8 \) for \( x \) in equation (1) to find \( y \).

\[-520x + 600y = 0 \quad (1)\]

\[-520(y + 8) + 600y = 0\]

\[-520y - 4160 + 600y = 0\]

\[80y = 4160\]

\[y = 52\]

Since \( y = 52 \) and \( x = y + 8 \), \( x = 60 \).
Step 5  State the answer.

The train’s speed is 60 mph, the truck’s speed is 52 mph.

Step 6  Check.

\[ 60 = 52 + 8 \]

It would take the train 10 hours to travel 600 miles at 60 mph, which is the same amount of time it would take the truck to travel 520 miles at 52 mph.

The answer is correct.
Objective 5

Solve problems with three variables by using a system of three equations.
EXAMPLE 5

A department store has three kinds of perfume; cheap, better, and best. It has 10 more bottles of the cheap than the better, and 3 fewer bottles of the best than the better. Each bottle of the cheap costs $8, better costs $15, and best costs $32. The total value of the all the perfume is $589. How many bottles of each are there?

**Step 1  Read** the problem. There are 3 unknowns.

**Step 2  Assign variables.**

Let $x =$ the number of bottles of cheap at $8

$y =$ the number of better at $15, and

$z =$ the number of best at $32.
continued

**Step 3** Write a system of equations.

There are 10 more bottles of cheap than better, so
\[ x = y + 10. \quad (1) \]

There are 3 fewer bottles of best than better, so
\[ z = y - 3. \quad (2) \]

The total value is $589, so
\[ 8x + 15y + 32z = 589. \quad (3) \]
continued

**Step 4 Solve.**

Substitute \( y + 10 \) for \( x \) and \( y - 3 \) for \( z \) in equation (3) to find \( y \).

\[
8(y + 10) + 15y + 32(y - 3) = 589 \\
8y + 80 + 15y + 32y - 96 = 589 \\
55y - 16 = 589 \\
55y = 605 \\
y = 11
\]

Since \( y = 11 \), \( x = y + 10 = 21 \) and \( z = y - 3 = 8 \).
continued

Step 5  State the answer.

There are 21 bottles of the cheap perfume, 11 of the better, and 8 of the best.

Step 6  Check.

\[ 21(8) + 11(15) + 8(32) = 589 \]

The answer is correct.
EXAMPLE 6

A paper mill makes newsprint, bond, and copy machine paper. Each ton of newsprint requires 3 tons of recycled paper and 1 ton of wood pulp. Each ton on bond requires 2 tons of recycled paper, and 4 tons of wood pulp, and 3 tons of rags. A ton of copy machine paper requires 2 tons of recycled paper, 3 tons of wood pulp, and 2 tons of rags. The mill has 4200 tons of recycled paper, 5800 tons of wood pulp, and 3900 tons of rags. How much of each kind of paper can be made from these supplies?

*Step 1 Read* the problem.
continued

**Step 2 Assign variables.**

Let $x =$ the number of tons of newsprint

$y =$ the number of tons of bond, and

$z =$ the number of tons of copy machine paper.

**Step 3 Write a system of equations.**

\[ 3x + 2y + 2z = 4200 \quad (1) \]
\[ x + 4y + 3z = 5800 \quad (2) \]
\[ 3y + 2z = 3900 \quad (3) \]
Step 4 Solve the system

\[3x + 2y + 2z = 4200\]
\[x + 4y + 3z = 5800\]
\[3y + 2z = 3900\]

to find \(x = 400, \ y = 900, \) and \(z = 600.\)
continued

Step 5  State the answer.

The paper mill can make 400 tons of newsprint, 900 tons of bond, and 600 tons of copy machine paper.

Step 6  Check that these values satisfy the conditions of the problem.

The answer is correct.
Solving Systems of Linear Equations by Matrix Methods

1. Define a matrix.
2. Write the augmented matrix of a system.
3. Use row operations to solve a system with two equations.
4. Use row operations to solve a system with three equations.
5. Use row operations to solve special systems.
Objective

Define a matrix.
A **matrix** is an ordered array of numbers.

The numbers are called **elements** of the matrix. Matrices are named according to the number of **rows** and **columns** they contain.

The number of rows followed by the number of columns give the **dimensions** of the matrix.
A square matrix is a matrix that has the same number of rows as columns.
Objective 2

Write the augmented matrix of a system.
An **augmented matrix** has a vertical bar that separates the columns of the matrix into two groups.

\[ x - 3y = 1 \]
\[ 2x + y = -5 \]

\[
\begin{bmatrix}
1 & -3 & 1 \\
2 & 1 & -5
\end{bmatrix}
\]
Matrix Row Operations

1. Any two rows of the matrix may be interchanged.
2. The elements of any row may be multiplied by any nonzero real number.
3. Any row may be changed by adding to the elements of the row the product of a real number and the corresponding elements of another row.
Examples of Row Operations

Interchanged any two rows.

\[
\begin{bmatrix}
2 & 3 & 9 \\
4 & 8 & -3 \\
1 & 0 & 7
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
1 & 0 & 7 \\
4 & 8 & -3 \\
2 & 3 & 9
\end{bmatrix}
\]

Interchange row 1 and row 3.

Multiply any row by a nonzero real number.

\[
\begin{bmatrix}
2 & 3 & 9 \\
4 & 8 & -3 \\
1 & 0 & 7
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
6 & 9 & 27 \\
4 & 8 & -3 \\
1 & 0 & 7
\end{bmatrix}
\]

Multiply row 1 by 3.
Examples of Row Operations (continued)

Multiply a row by a nonzero number and add to another row.

Multiply row 3 by $-2$; add them to the corresponding numbers in row 1.

\[
\begin{bmatrix}
2 & 3 & 9 \\
4 & 8 & -3 \\
1 & 0 & 7
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
0 & 3 & -5 \\
4 & 8 & -3 \\
1 & 0 & 7
\end{bmatrix}
\]
Objective 3

Use row operations to solve a system with two equations.
Row operations can be used to rewrite a matrix until it is the matrix of a system whose solution is easy to find. The goal is a matrix in the form

\[
\begin{bmatrix}
1 & a & b \\
0 & 1 & c
\end{bmatrix}
\]

for systems with two and three equations.

A matrix written as shown above with a diagonal of ones, is said to be in **row echelon form**.
EXAMPLE 1

Use row operations to solve the system.

\[x - 2y = 9\]
\[3x + y = 13\]

Write the augmented matrix of the system.

\[
\begin{bmatrix}
1 & -2 & | & 9 \\
3 & 1 & | & 13
\end{bmatrix}
\]

Use row operations to change the matrix into one that leads to a system that is easy to solve.

It is best to work by columns.
continued

\[-3R_1 + R_2\]

\[
\begin{bmatrix}
1 & -2 & 9 \\
3 & 1 & 13 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & -2 & 9 \\
3+(1)(-3) & 1+(-2)(-3) & 13+(-3)(9) \\
\end{bmatrix}
\]

Original number from row 2

-3 times the number from row 1

\[
\begin{bmatrix}
1 & -2 & 9 \\
0 & 7 & -14 \\
\end{bmatrix}
\]

\[
\frac{1}{7} R_2
\]

\[
\begin{bmatrix}
1 & -2 & 9 \\
0 & 1 & -2 \\
\end{bmatrix}
\]

\[x - 2y = 9\]

\[3x + y = 13\]
The matrix gives the system: \[ x - 2y = 9 \]
\[ y = -2 \]

Substitute \(-2\) for \(y\) in the first equation.
\[ x - 2(-2) = 9 \]
\[ x + 4 = 9 \]
\[ x = 5 \]

The solution set is \(\{(5, -2)\}\).
Objective 4

Use row operations to solve a system with three equations.
EXAMPLE 2

Use row operations to solve the system.

\[
\begin{align*}
2x - y + z &= 7 \\
x - 3y - z &= 7 \\
-x + y - 5z &= -9
\end{align*}
\]

Interchange rows 1 and 2.

\[
\begin{align*}
x - 3y - z &= 7 \\
2x - y + z &= 7 \\
-x + y - 5z &= -9
\end{align*}
\]

Write the augmented matrix of the system.

\[
\begin{bmatrix}
1 & -3 & -1 & 7 \\
2 & -1 & 1 & 7 \\
-1 & 1 & -5 & -9
\end{bmatrix}
\]
Write the augmented matrix of the system.

\[
\begin{bmatrix}
1 & -3 & -1 & 7 \\
2 & -1 & 1 & 7 \\
-1 & 1 & -5 & -9
\end{bmatrix}
\]

\[-2R_1 + R_2 \]

\[
\begin{bmatrix}
1 & -3 & -1 & 7 \\
0 & 5 & 3 & -7 \\
-1 & 1 & -5 & -9
\end{bmatrix}
\]

\[R_1 + R_3 \]

\[
\begin{bmatrix}
1 & -3 & -1 & 7 \\
0 & 5 & 3 & -7 \\
0 & -2 & -6 & -2
\end{bmatrix}
\]
continued

\[
\begin{bmatrix}
1 & -3 & -1 & 7 \\
0 & 5 & 3 & -7 \\
0 & -2 & -6 & -2
\end{bmatrix} \quad \frac{1}{5} R_2
\]

\[
\begin{bmatrix}
1 & -3 & -1 & 7 \\
0 & 1 & \frac{3}{5} & \frac{-7}{5} \\
0 & -2 & -6 & -2
\end{bmatrix}
\]

\[2R_2 + R_3\]

\[
\begin{bmatrix}
1 & -3 & -1 & 7 \\
0 & 1 & \frac{3}{5} & \frac{-7}{5} \\
0 & 0 & \frac{-24}{5} & \frac{-24}{5}
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & -3 & -1 & 7 \\
0 & 1 & \frac{3}{5} & \frac{-7}{5} \\
0 & 0 & \frac{-24}{5} & \frac{-24}{5}
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & -3 & -1 & 7 \\
0 & 1 & \frac{3}{5} & \frac{-7}{5} \\
0 & 0 & 1 & 1
\end{bmatrix}
\]
This matrix gives the system

\[
\begin{align*}
  x - 3y - z &= 7 \\
  y + \frac{3}{5}z &= -\frac{7}{5} \\
  z &= 1
\end{align*}
\]

Substitute 1 for \(z\) in the second equation.

\[
y + \frac{3}{5}(1) = -\frac{7}{5}
\]

\[
y = -2
\]
continued

Substitute $-2$ for $y$ and $1$ for $z$ in the first equation.

\[
x - 3y - z = 7
\]

\[
x - 3(-2) - 1 = 7
\]

\[
x + 5 = 7
\]

\[
x = 2
\]

The solution set is $\{(2, -2, 1)\}$.
Objective 5

Use row operations to solve special systems.
EXAMPLE 3

Use row operations to solve each system.

\[ x - y = 2 \]
\[ -2x + 2y = 2 \]

a. Write the augmented matrix.

\[
\begin{bmatrix}
1 & -1 & | & 2 \\
-2 & 2 & | & 2 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & -1 & | & 2 \\
-2 & 2 & | & 2 \\
\end{bmatrix} 2R_1 + R_2 
\begin{bmatrix}
1 & -1 & | & 2 \\
0 & 0 & | & 6 \\
\end{bmatrix}
\]
continued

The matrix gives the system: \[ x - y = 2 \]
\[ 0 = 6 \]

The false statement indicates that the system is inconsistent and has no solution.

The solution set is \( \emptyset \).
continued

b. \( x - y = 2 \)
\[ -2x + 2y = -4 \]

Write the augmented matrix.

\[
\begin{bmatrix}
1 & -1 & | & 2 \\
-2 & 2 & | & -4
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & -1 & | & 2 \\
-2 & 2 & | & -4
\end{bmatrix} \quad 2R_1 + R_2 \quad \begin{bmatrix}
1 & -1 & | & 2 \\
0 & 0 & | & 0
\end{bmatrix}
\]
continued

The matrix gives the system: \[ x - y = 2 \]
\[ 0 = 0 \]

The true statement indicates that the system has dependent equations.

The solution set is \( \{(x, y) | x - y = 2\} \).