5.1 Integer Exponents and Scientific Notation

1. Use the product rule for exponents.
2. Define 0 and negative exponents.
3. Use the quotient rule for exponents.
4. Use the power rules for exponents.
5. Simplify exponential expressions.
6. Use the rules for exponents with scientific notation.
Objective 1

Use the product rule for exponents.
### Product Rule for Exponents

If \( m \) and \( n \) are natural numbers and \( a \) is any real number, then

\[
a^m \cdot a^n = a^{m+n}.
\]

That is, when multiplying powers of like bases, keep the same base and add the exponents.
EXAMPLE 1

Apply the product rule for exponents.

a. \( m^8 \cdot m^6 = m^{8+6} = m^{14} \)

b. \( m^5 \cdot p^4 \) Can not be simplified further because the bases \( m \) and \( p \) are not the same. The product rule does not apply.

c. \( (−5p^4)(−9p^5) = (−5)(−9)(p^4p^5) = 45p^{4+5} = 45p^9 \)

d. \( (−3x^2y^3)(7xy^4) = (−3)(7)x^2xy^3y^4 = −21x^{2+1}y^{3+4} = −21x^3y^7 \)
Objective 2

Define 0 and negative exponents.
Zero Exponent

If \( a \) is any nonzero real number, then

\[ a^0 = 1. \]
EXAMPLE 2

Evaluate.

a. \( 29^0 = 1 \)

b. \((-29)^0 = 1\)

c. \(-29^0 = -(29^0) = -1\)

d. \(8^0 - 15^0 = 1 - 1 = 0\)
Negative Exponent

For any natural number $n$ and any nonzero real number $a$, 
\[
a^{-n} = \frac{1}{a^n}.
\]

**CAUTION** A negative exponent does not indicate a negative number; negative exponents lead to reciprocals. For example,
\[
3^{-2} = \frac{1}{3^{-2}} = \frac{1}{9} \quad \text{Not negative}
\]
\[
-3^{-2} = - \frac{1}{3^{-2}} = -\frac{1}{9} \quad \text{Negative}
\]
EXAMPLE 3

Write each exponential with only positive exponents.

a. \(6^{-5} = \frac{1}{6^5}\)

b. \((2x)^{-4}, x \neq 0 = \frac{1}{2x^4}, x \neq 0\)

c. \(-7p^{-4}, p \neq 0 = -7\left(\frac{1}{p^4}\right) = -\frac{7}{p^4}, p \neq 0\)

d. Evaluate \(4^{-1} - 2^{-1}\).

\[4^{-1} - 2^{-1} = \frac{1}{4} - \frac{1}{2} = \frac{1}{4} - \frac{2}{4} = -\frac{1}{4}\]
EXAMPLE 4

Evaluate.

a. \[ \frac{1}{4^{-3}} \]  
b. \[ \frac{3^{-3}}{9^{-1}} \]

a. \[
\frac{1}{4^{-3}} = \frac{1}{\frac{1}{4^3}} = 1 \div \frac{1}{4^3} = 1 \cdot \frac{4^3}{1} = 4^3 = 64
\]

b. \[
\frac{3^{-3}}{9^{-1}} = \frac{3^{-3}}{\frac{1}{9}} = \frac{1}{3^3} \div \frac{1}{9} = \frac{1}{3^3} \cdot \frac{9}{1} = \frac{1}{27} \cdot \frac{9}{1} = \frac{9}{27} = \frac{1}{3}
\]
<table>
<thead>
<tr>
<th>Special Rules for Negative Exponents</th>
</tr>
</thead>
<tbody>
<tr>
<td>If $a \neq 0$ and $b \neq 0$, then</td>
</tr>
</tbody>
</table>
| \[
| \frac{1}{a^{-n}} = a^n \quad \text{and} \quad \frac{a^{-n}}{b^{-m}} = \frac{b^m}{a^n}.
|\] |
Objective 3

Use the quotient rule for exponents.
Quotient Rule for Exponents

If \( a \) is any nonzero real number and \( m \) and \( n \) are integers, then

\[
\frac{a^m}{a^n} = a^{m-n}.
\]

That is, when dividing powers of like bases, keep the same base and subtract the exponent of the denominator from the exponent of the numerator.
EXAMPLE 5

Apply the quotient rule, if possible, and write each result with only positive exponents.

a. \( \frac{m^8}{m^{13}} = m^{8-13} = m^{-5} = \frac{1}{m^5}, \quad m \neq 0 \)

b. \( \frac{5^{-6}}{5^{-8}} = 5^{-6-(-8)} = 5^{-6+8} = 5^2, \quad \text{or} \ 25. \)

c. \( \frac{x^3}{y^5}, \quad y \neq 0 \) Can not be simplified because the bases \( x \) and \( y \) are different. The quotient rule does not apply.
Objective 4

Use the power rules for exponents.
Power Rule for Exponents

If $a$ and $b$ are real numbers and $m$ and $n$ are integers, then

\[ a^m \, ^n = a^{mn}, \quad ab^m = a^m b^m, \]

and

\[ \left( \frac{a}{b} \right)^m = \frac{a^m}{b^m} \quad (b \neq 0). \]

That is,

a) To raise a power to a power, multiply exponents.

b) To raise a product to a power, raise each factor to that power.

c) To raise a quotient to a power, raise the numerator and the denominator to that power.
EXAMPLE 6

Simplify using the power rules.

a. \( r^5 \cdot 4 = r^{5 \cdot 4} = r^{20} \)

b. \( -3 \cdot y^5 \cdot 2 = -3^2 \cdot y^{5 \cdot 2} = 9 \cdot y^{10} \)

c. \( \left( \frac{3}{4} \right)^3 = \frac{3^3}{4^3} = \frac{27}{64} \)
More Special Rules for Negative Exponents

If $a \neq 0$ and $b \neq 0$ and $n$ is an integer, then

$$a^{-n} = \left(\frac{1}{a}\right)^n$$
and

$$\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n.$$  

That is, any nonzero number raised to the negative $n$th power is equal to the reciprocal of that number raised to the $n$th power.
EXAMPLE 7

Write with only positive exponents and then evaluate.

\[
\left( \frac{2}{3} \right)^{-4} = \left( \frac{3}{2} \right)^{4} = \frac{3^4}{2^4} = \frac{81}{16}
\]
Definition and Rules for Exponents

For all integers \( m \) and \( n \) and all real numbers \( a \) and \( b \), the following rules apply.

**Product Rule** \[ a^m \cdot a^n = a^{m+n} \]

**Quotient Rule** \[ \frac{a^m}{a^n} = a^{m-n} \quad a \neq 0 \]

**Zero Exponent** \[ a^0 = 1 \quad a \neq 0 \]
Definition and Rules for Exponents (continued)

Negative Exponent
\[a^{-n} = \frac{1}{a^n} \quad (a \neq 0)\]

Power Rules
\[a^m \cdot n = a^{mn} \quad \quad ab^m = a^m b^m\]
\[\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m} \quad b \neq 0\]

Special Rules
\[\frac{1}{a^{-n}} = a^n \quad a \neq 0 \quad \quad \frac{a^{-n}}{b^{-m}} = \frac{b^m}{a^n} \quad a, b \neq 0\]
\[a^{-n} = \left(\frac{1}{a}\right)^n \quad a \neq 0 \quad \left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n \quad a, b \neq 0\]
Objective 5

Simplify exponential expressions.
EXAMPLE 8

Simplify so that no negative exponents are in the final result. Assume that all variables represent nonzero real numbers.

a. $4^2^{-5} = 4^{2(-5)} = 4^{-10} = \frac{1}{4^{10}}$

b. $x^{-4} \cdot x^{-6} \cdot x^8 = x^{-4+(-6)+8} = x^{-2} = \frac{1}{x^2}$

c. \[
\frac{m^2 n^{-2}}{m^{-3} n} = \frac{m^2}{m^{-3}} \cdot \frac{n^{-2}}{n} = \frac{m^{-4} n^{-2}}{m^{-3} n} = \frac{m^{-4}}{m^{-3}} \cdot \frac{n^{-2}}{n} = m^{-4-(-3)} n^{-2-1} = m^{-4+3} n^{-3} = m^{-1} n^{-3} = \frac{1}{mn^3}
\]
d. \( \left( \frac{2y}{x^3} \right)^2 \left( \frac{4y}{x} \right)^{-1} \)

\[
\begin{align*}
&= \frac{2^2 y^2}{x^6} \cdot \frac{4^{-1} y^{-1}}{x^{-1}} \\
&= 2^2 4^{-1} y^1 \\
&= \frac{y}{x^5} \\
&= \frac{2^2 y}{4x^5} = \frac{y}{x^5}
\end{align*}
\]

Combination of rules
Objective 6

Use the rules for exponents with scientific notation.
Scientific Notation

A number is written in **scientific notation** when it is expressed in the form

$$a \times 10^n$$

where $1 \leq |a| < 10$ and $n$ is an integer.
Converting to Scientific Notation

**Step 1**  **Position the decimal point.** Place a caret, ^, to the right of the first nonzero digit, where the decimal point will be placed.

**Step 2**  **Determine the numeral for the exponent.** Count the number of digits from the decimal point to the caret. This number gives the absolute value of the exponent on 10.

**Step 3**  **Determine the sign for the exponent.** Decide whether multiplying by $10^n$ should make the result of Step 1 greater or less. The exponent should be positive to make the result greater; it should be negative to make the result less.
EXAMPLE 9

Write the number in scientific notation.

29,800,000

Step 1 Place a caret to the right of the 2 (the first nonzero digit) to mark the new location of the decimal point.

Step 2 Count from the decimal point, which is understood to be after the caret.

29,800,000 = 2.9,800,000.

Step 3 Since 2.98 is to be made greater, the exponent on 10 is positive.

29,800,000 = 2.98 \times 10^7
EXAMPLE 10

Write the number in scientific notation.

\[ 0.0000000503 \]

**Step 1** Place a caret to the right of the 5 (the first nonzero digit) to mark the new location of the decimal point.

**Step 2** Count from the decimal point 8 places, which is understood to be after the caret.

\[ 0.0000000503 = 0.0000000503 \]

**Step 3** Since 5.03 is to be made less, the exponent 10 is negative.

\[ 0.0000000503 = 5.03 \times 10^{-8} \]
Converting from Scientific Notation

Multiplying a number by a positive power of 10 makes the number greater, so move the decimal point to the right if $n$ is positive in $10^n$.

Multiplying a number by a negative power of 10 makes the number less, so move the decimal point to the left if $n$ is negative.

If $n$ is 0, leave the decimal point where it is.
EXAMPLE 11

Write each number in standard notation.

a. \(2.51 \times 10^3 = 2.510. = 2510\)
   Move the decimal 3 places to the right.

b. \(-6.8 \times 10^{-5} = -0.00006.8 = -0.000068\)
   Move the decimal 4 places to the left.
EXAMPLE 12

Evaluate \[ \frac{200,000 \times 0.0003}{0.06} \].

\[
= \frac{2 \times 10^5 \times 3 \times 10^{-4}}{6 \times 10^{-2}}
\]

\[
= \frac{2 \times 3 \times 10^1}{6 \times 10^{-2}}
\]

\[
= \frac{2 \times 3}{6} \times 10^3 = 1 \times 10^3 = 1000
\]
EXAMPLE 13

The distance to the sun is $9.3 \times 10^7$ mi. How long would it take a rocket traveling at $3.2 \times 10^3$ mph to reach the sun?

\[ d = rt, \text{ so} \]
\[ t = \frac{d}{r} \]
\[ = \frac{9.3 \times 10^7}{3.2 \times 10^3} = \frac{9.3}{3.2} \times 10^{7-3} \approx 2.9 \times 10^4. \]

It would take approximately $2.9 \times 10^4$. 
Adding and Subtracting Polynomials

1. Know the basic definitions for polynomials.
2. Find the degree of a polynomial.
3. Add and subtract polynomials.
Objective

Know the basic definitions for polynomials.
A **term** is a number, a variable, or the product or quotient of a number and one or more variables raised to powers.

\[ 4x, \quad \frac{1}{2}m^5 \quad \text{or} \quad \left( \frac{m^5}{2} \right), \quad -7z^9, \quad 6x^2z, \quad \frac{5}{3x^2}, \quad \text{and} \quad 9. \]

The number in the product is called the **numerical coefficient**, or just the **coefficient**.

- \( 8k^3 \) \quad 8 \quad \text{is the coefficient}
- \( -4p^5 \) \quad -4 \quad \text{is the coefficient}
An algebraic expression is any combination of variables or constants (numerical values) joined by the basic operations of addition, subtraction, multiplication, and division (expect by 0), or raising to powers or taking roots, formed according to the rules of algebra.
Polynomial

A **polynomial** is a term or a finite sum of terms in which all variables have whole number exponents and no variables appear in denominators.
Polynomials

\[3x - 5, \quad 4m^3 - 5m^2 p + 8, \quad \text{and} \quad -5t^2 s^3\]

Not Polynomials

\[x^{-1} + 3x^{-2}, \quad \sqrt{9 - x}, \quad \text{and} \quad \frac{1}{x}\]
A polynomial in one variable is written in **descending powers** of the variable if the exponents on the variable decrease from left to right.

\[ x^5 - 6x^2 + 12x - 5 \]
EXAMPLE 1

Write the polynomial in descending powers of the variable.

\[-3z^4 + 2z^3 + z^5 - 6z\]

The largest exponent is 5, it would be the first term.

\[z^5 - 3z^4 + 2z^3 - 6z\]
Some polynomials with a specific number of terms are so common that they are given special names.

**Trinomial**: has exactly three terms

**Binomial**: has exactly two terms

**Monomial**: has only one term

<table>
<thead>
<tr>
<th>Type of Polynomial</th>
<th>Examples</th>
</tr>
</thead>
<tbody>
<tr>
<td>Monomial</td>
<td>$5x$, $7m^9$, $-8$, $x^2y^2$</td>
</tr>
<tr>
<td>Binomial</td>
<td>$3x^2 - 6$, $11y + 8$, $5a^2b + 3a$</td>
</tr>
<tr>
<td>Trinomial</td>
<td>$y^2 + 11y + 6$, $8p^3 - 7p + 2m$, $-3 + 2k^5 + 9z^4$</td>
</tr>
<tr>
<td>None of these</td>
<td>$p^3 - 5p^2 + 2p - 5$, $-9z^3 + 5c^2 + 2m^5 + 11r^2 - 7r$</td>
</tr>
</tbody>
</table>
Objective 2

Find the degree of a polynomial.
The **degree of a term** with one variable is the exponent on the variable.

The degree of $2x^3$ is 3.
The degree of $-x^4$ is 4.
The degree of $17x$ is 1.

The greatest degree of any term in a polynomial is called the **degree of the polynomial**.
The table shows several polynomials and their degrees.

<table>
<thead>
<tr>
<th>Polynomial</th>
<th>Degree</th>
</tr>
</thead>
<tbody>
<tr>
<td>$9x^2 - 5x + 8$</td>
<td>2</td>
</tr>
<tr>
<td>$17m^9 + 18m^{14} - 9m^3$</td>
<td>14</td>
</tr>
<tr>
<td>$5x$</td>
<td>1, because $5x = 5x^1$</td>
</tr>
<tr>
<td>$-2$</td>
<td>0, because $-2 = -2x^0$</td>
</tr>
<tr>
<td></td>
<td>(Any nonzero constant has degree 0.)</td>
</tr>
<tr>
<td>$5a^2b^5$</td>
<td>7, because $2 + 5 = 7$</td>
</tr>
<tr>
<td>$x^3y^9 + 12xy^4 + 7xy$</td>
<td>12, because the degree of the terms are 12, 5, and 2, and 12 is the greatest.</td>
</tr>
</tbody>
</table>
Objective 3

Add and subtract polynomials.
EXAMPLE 2

Combine like terms.

a. $2z^4 + 3x^4 + z^4 - 9x^4$

b. $3t + 4r - 4t - 8r$

c. $5x^2z - 3x^3z^2 + 8x^2z + 12x^3z^2$

a. $2z^4 + 3x^4 + z^4 - 9x^4 = 2z^4 + z^4 + 3x^4 - 9x^4$

$= 3z^4 - 6x^4$

b. $3t + 4r - 4t - 8r = 3t - 4t + 4r - 8r$

$= -t - 4r$

c. $5x^2z - 3x^3z^2 + 8x^2z + 12x^3z^2$

$= 5x^2z + 8x^2z - 3x^3z^2 + 12x^3z^2$

$= 13x^2z + 9x^3z^2$
Adding Polynomials

To add two polynomials, combine like terms.
EXAMPLE 3

Add.

a. \((-5p^3 + 6p^2) + (8p^3 - 12p^2)\)

Use commutative and associative properties to rearrange the polynomials so that like terms are together. Then use the distributive property to combine like terms.

\[
(-5p^3 + 6p^2) + (8p^3 - 12p^2) = -5p^3 + 8p^3 + 6p^2 - 12p^2 = 3p^3 - 6p^2
\]
You can add polynomials vertically by placing like terms in columns.

\[ \begin{align*}
\text{b.} & & -6r^5 + 2r^3 - r^2 \\
& & 8r^5 - 2r^3 + 5r^2 \\
& & \frac{8r^5 - 2r^3 + 5r^2}{2r^5 + 4r^2}
\end{align*} \]

The solution is \(2r^5 + 4r^2\)
Subtracting Polynomials

To subtract two polynomials, add the first polynomial and the negative of the second polynomial.
EXAMPLE 4

Subtract

a. \((p^4 + p^3 + 5) - (3p^4 + 5p^3 + 2)\)

Change every sign in the second polynomial and add.

\[
\begin{align*}
(p^4 + p^3 + 5) - (3p^4 + 5p^3 + 2) &= p^4 + p^3 + 5 - 3p^4 - 5p^3 - 2 \\
&= p^4 - 3p^4 + p^3 - 5p^3 + 5 - 2 \\
&= -2p^4 - 4p^3 + 3
\end{align*}
\]
To subtract vertically, write the first polynomial above the second, lining up like terms in columns.

Change all the signs in the second polynomial and add.

b. \[ 2k^3 - 3k^2 - 2k + 5 \]
\[ 4k^3 + 6k^2 - 5k + 8 \]

\[ \underline{2k^3 - 3k^2 - 2k + 5} \]
\[ -4k^3 - 6k^2 + 5k - 8 \]
\[ \underline{2k^3 - 9k^2 + 3k - 3} \]
5.3 Polynomial Functions, Graphs and Composition

1. Recognize and evaluate polynomial functions.
2. Use a polynomial function to model data.
3. Add and subtract polynomial functions.
4. Find the composition of functions.
5. Graph basic polynomial functions.
Objective

Recognize and evaluate polynomial functions.
Polynomial Function

A polynomial function of degree $n$ is defined by

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0,$$

for real numbers $a_n, a_{n-1}, \ldots a_1$, and $a_0$, where $a_n \neq 0$ and $n$ is a whole number.
EXAMPLE 1

Let \( f(x) = -x^2 + 5x - 11 \). Find \( f(-4) \)

\[
f(-4) = -x^2 + 5x - 11 \\
= -(-4)^2 + 5(-4) - 11 \quad \text{Substitute } -4 \text{ for } x. \\
= -16 - 20 - 11 \quad \text{Order of operations} \\
= -47 \quad \text{Subtract.}
\]

Read this as “\( f \) of negative 4,” not \( f \) times negative 4.”
Objective 2

Use a polynomial function to model data.
EXAMPLE 2

The number of U.S. households estimated to see and pay at least one bill on-line each month during the years 2000 through 2006 can be modeled by the polynomial function defined by

\[ P(x) = 0.808x^2 + 2.625x + 0.502. \]

where \( x = 0 \) corresponds to the year 2000, \( x = 1 \) corresponds to the year 2001, and so on and \( P(x) \) is in millions. Use the function to approximate the number of households that paid at least one bill on-line each month in 2006.

\[
P(x) = 0.808x^2 + 2.625x + 0.502
\]

\[
= 0.808(6)^2 + 2.625(6) + 0.502
\]

\[
= 45.34 \text{ million households}
\]
Objective 3

Add and subtract polynomial functions.
The operations of addition, subtraction, multiplication, and division are also defined for functions.
For example, businesses use the equation “profit equals revenue minus cost,” written in function notation as

\[ P(x) = R(x) - C(x) \]

where \( x \) is the number of items produced and sold.
### Adding and Subtracting Functions

If \( f(x) \) and \( g(x) \) define functions, then

\[
(f + g)(x) = f(x) + g(x)
\]

**Sum function**

and

\[
(f - g)(x) = f(x) - g(x).
\]

**Difference function**

In each case, the domain of the new function is the intersection of the domains of \( f(x) \) and \( g(x) \).
EXAMPLE 3

Let \( f(x) = 3x^2 + 8x - 6 \) and \( g(x) = -4x^2 + 4x - 8 \). Find each function.

a. \( (f + g)(x) \)

\[
(f + g)(x) = f(x) + g(x) = (3x^2 + 8x - 6) + (-4x^2 + 4x - 8) = -x^2 + 12x - 14
\]

b. \( (f - g)(x) \)

\[
(f - g)(x) = f(x) - g(x) = (3x^2 + 8x - 6) - (-4x^2 + 4x - 8) = 3x^2 + 8x - 6 + 4x^2 - 4x + 8 = 7x^2 + 4x + 2
\]
**EXAMPLE 4**

Let \( f(x) = 18x^2 - 24x \) and \( g(x) = 3x \), find each of the following.

a. \((f + g)(x)\) and \((f + g)(-1)\)

\[
(f + g)(x) = f(x) + g(x)
= 18x^2 - 24x + 3x
= 18x^2 - 21x
\]

\[
(f + g)(-1) = f(x) + g(x)
= [18(-1)^2 - 24(-1)] + 3(-1)
= [18 + 24] - 3 = 39
\]

b. \((f - g)(x)\) and \((f - g)(1)\)
continued

b. \((f - g)(x)\) and \((f - g)(1)\)

\[(f - g)(x) = f(x) - g(x)\]
\[= 18x^2 - 24x - 3x\]
\[= 18x^2 - 27x\]

\[(f - g)(1) = f(x) - g(x)\]
\[= [18(1)^2 - 24(1)] - 3(1)\]
\[= [18 - 24] - 3\]
\[= -9\]
Objective 4

Find the composition of functions.
Composition of Functions

If \( f \) and \( g \) are functions, then the **composite function**, or **composition**, of \( g \) and \( f \) is defined by

\[
g \circ f \ (x) = g(f(x))
\]

for all \( x \) in the domain of \( f \) such that \( f(x) \) is in the domain of \( g \).
EXAMPLE 5

Let \( f(x) = x - 4 \) and \( g(x) = x^2 \)

Find \( f \circ g \ (3) \).

By definition \( f \circ g \ (3) = f(g(3)) \)

\[
= f(3^2)
\]

\[
= f(9)
\]

\[
= 9 - 4
\]

\[
= 5
\]

Evaluate the “inside” function value first.

Now evaluate the “outside” function.
EXAMPLE 6

Let \( f(x) = 3x + 6 \) and \( g(x) = x^3 \)

Find the following.

a. \( f \circ g \)(2)

\[
f \circ g \( (2) = f(2^3) = f(8) = 3(8) + 6 = 30
\]

b. \( g \circ f \)(x)

\[
g \circ f \( (x) = g(3x + 6) = (3x + 6)^3
\]
Objective 5

Graph basic polynomial functions.
Identity Function

Defined as \( f(x) = x \).

The domain (set of \( x \)-values) is all real numbers, \((-\infty, \infty)\).

The range (set of \( y \)-values) is also \((-\infty, \infty)\).

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) = x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>-2</td>
</tr>
<tr>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
</tr>
</tbody>
</table>
Squaring Function

Defined as \( f(x) = x^2 \).

The domain is all real numbers, \((-\infty, \infty)\).

The range is \([0, \infty)\).

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) = x^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>4</td>
</tr>
<tr>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
</tr>
</tbody>
</table>
Cubing Function

Defined as $f(x) = x^3$.

The domain and range are both $(-\infty, \infty)$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x) = x^3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>-8</td>
</tr>
<tr>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>8</td>
</tr>
</tbody>
</table>
EXAMPLE 7

Graph $f(x) = -2x^2$. Give the domain and range.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x) = -2x^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>-8</td>
</tr>
<tr>
<td>-1</td>
<td>-2</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>-2</td>
</tr>
<tr>
<td>2</td>
<td>-8</td>
</tr>
</tbody>
</table>

The **domain** is $(-\infty, \infty)$.

The **range** is $(-\infty, 0]$. 
5.4 Multiplying Polynomials

1. Multiply terms.
2. Multiply any two polynomials.
4. Find the product of the sum and difference of two terms.
5. Find the square of a binomial.
Objective 1

Multiply terms.
EXAMPLE 1

Find the product.

$$8k^3y(9ky)$$

$$= (8)(9)k^3 \cdot k^1 \cdot y^1 \cdot y^1$$

$$= 72k^{3+1}y^{1+1}$$

$$= 72k^4y^2$$
Objective 2

Multiply any two polynomials.
EXAMPLE 2

Find each product.

a. \(-2r(9r - 5) = -2r(9r) - 2r(-5)\)

\[= -18r^2 + 10r\]
continued

b. \((2k - 5m)(3k + 2m) = (2k - 5m)(3k) + (2k - 5m)(2m)\)

\[= 2k(3k) + (-5m)(3k) + (2k)(2m) + (-5m)(2m)\]

\[= 6k^2 - 15km + 4km - 10m^2\]

\[= 6k^2 - 11km - 10m^2\]
EXAMPLE 3

Find each product.

a. \((4x - 3y)(3x - y)\)

\[
\begin{align*}
4x & - 3y \\
3x & - y \\
\hline
-4xy & + 3y^2
\end{align*}
\]

\[
12x^2 - 9xy
\]

\[
12x^2 - 13xy + 3y^2
\]

\[\text{Multiply } -y(4x - 3y)\]

\[\text{Multiply } 3x(4x - 3y)\]

\[\text{Combine like terms.}\]
Find each product.

b. \((5a^3 - 6a^2 + 2a - 3)(2a - 5)\)

\[
\begin{align*}
5a^3 - 6a^2 + 2a - 3 & \quad \quad 2a - 5 \\
\hline
2a - 5 &
\end{align*}
\]

\[-25a^3 + 30a^2 - 10a + 15\]

\[
\begin{align*}
10a^4 - 12a^3 + 4a^2 - 6a & \quad \quad 10a^4 - 37a^3 + 34a^2 - 16a + 15 \\
\end{align*}
\]

Combine like terms.
Objective 3

Multiply binomials.
When working with polynomials, the products of two binomials occurs repeatedly. There is a shortcut method for finding these products.

**First Terms**

**Outer Terms**

**Inner Terms**

**Last terms**

**CAUTION** The FOIL method is an extension of the distributive property, and the acronym “**FOIL**” applies only to multiplying two binomials.
EXAMPLE 4

Use the FOIL method to find each product.

a. \((5r - 3)(2r - 5)\)

\[
\begin{align*}
\text{F} & \quad \text{O} & \quad \text{I} & \quad \text{L} \\
= (5r)(2r) & + (5r)(-5) & + (-3)(2r) & + (-3)(-5) \\
= 10r^2 - 25r & - 6r & + 15 \\
= 10r^2 - 31r + 15
\end{align*}
\]

b. \((4y - z)(2y + 3z)\)

\[
\begin{align*}
\text{F} & \quad \text{O} & \quad \text{I} & \quad \text{L} \\
= (4y)(2y) & + (4y)(3z) & + (-z)(2y) & + (-z)(3z) \\
= 8y^2 & + 12yz & - 2yz & - 3z^2 \\
= 8y^2 & + 10yz & - 3z^2
\end{align*}
\]
Objective 4

Find the product of the sum and difference of two terms.
Product of the Sum and Difference of Two Terms

The **product of the sum and difference of the two terms** $x$ and $y$ is the difference of the squares of the terms.

$$(x + y)(x - y) = x^2 - y^2$$
EXAMPLE 5

Find each product.

a. \((m + 5)(m - 5) = m^2 - 5^2\)
   \[= m^2 - 25\]

b. \((x - 4y)(x + 4y) = x^2 - (4y)^2\)
   \[= x^2 - 4^2y^2\]
   \[= x^2 - 16y^2\]

c. \(4y^2(y + 7)(y - 7) = 4y^2(y^2 - 49)\)
   \[= 4y^4 - 196y^2\]
Objective 5

Find the square of a binomial.
Square of a Binomial

The square of a binomial is the sum of the square of the first term, twice the product of the two terms, and the square of the last term.

\[(x + y)^2 = x^2 + 2xy + y^2\]
\[(x - y)^2 = x^2 - 2xy + y^2\]
EXAMPLE 6

Find each product.

a. \((t + 9)^2 = t^2 + 2 \cdot t \cdot 9 + 9^2\)
   \[= t^2 + 18t + 81\]

b. \((2m + 5)^2 = (2m)^2 + 2(2m)(5) + 5^2\)
   \[= 4m^2 + 20m + 25\]

c. \((3k - 2n)^2 = (3k)^2 - 2(3k)(2n) + (2n)^2\)
   \[= 9k^2 - 12kn + 4n^2\]
EXAMPLE 7

Find each product.

a. \[ (x - y) + z][(x - y) - z] = (x - y)^2 - z^2 \]
   \[ = x^2 - 2(x)(y) + y^2 - z^2 \]
   \[ = x^2 - 2xy + y^2 - z^2 \]

b. \[ (p + 2q)^3 = (p + 2q)^2(p + 2q) \]
   \[ = (p^2 + 4pq + 4q^2)(p + 2q) \]
   \[ = p^3 + 4p^2q + 4pq^2 + 2p^2q + 8pq^2 + 8q^3 \]
   \[ = p^3 + 6p^2q + 12pq^2 + 8q^3 \]
EXAMPLE 7

c. \((x + 2)^4\)

\[\begin{align*}
&= (x + 2)^2 (x + 2)^2 \\
&= (x^2 + 4x + 4) (x^2 + 4x + 4) \\
&= x^4 + 4x^3 + 4x^2 + 4x^3 + 16x^2 + 16x + 4x^2 + 16x + 16 \\
&= x^4 + 8x^3 + 24x^2 + 32x + 16
\end{align*}\]
Objective 6

Multiply polynomial functions.
**Multiplying Functions**

If $f(x)$ and $g(x)$ define functions, then

$$(fg)(x) = (f)x \cdot g(x).$$

**Product function**

The domain of the product function is the intersection of the domains of $f(x)$ and $g(x)$. 
EXAMPLE 8

For \( f(x) = 3x + 1 \) and \( g(x) = 2x - 5 \), find \( (fg)(x) \) and \( (fg)(2) \).

\[
(fg)(x) = f(x) \cdot g(x).
\]

\[
= (3x + 1)(2x - 5)
\]

\[
= 6x^2 - 15x + 2x - 5
\]

\[
= 6x^2 - 13x - 5
\]

Then

\[
(fg)(2) = 6(2)^2 - 13(2) - 5
\]

\[
= 24 - 26 - 5
\]

\[
= -7.
\]
Dividing Polynomials

1. Divide a polynomial by a monomial.
2. Divide a polynomial by a polynomial of two or more terms.
3. Divide polynomial functions.
Objective 1

Divide a polynomial by a monomial.
Dividing a Polynomial by a Monomial

To divide a polynomial by a monomial, divide each term in the polynomial by the monomial, and then write each quotient in lowest terms.
EXAMPLE 1

Divide.

a. \[
\frac{10x^2 - 25x + 35}{5}
\]

\[
\frac{10x^2 - 25x + 35}{5} = \frac{10x^2}{5} - \frac{25x}{5} + \frac{35}{5}
\]

\[
= 2x^2 - 5x + 7
\]

Check this answer by multiplying it by the divisor, 5.

\[
5 \cdot 2x^2 - 5x + 7 = 10x^2 - 25x + 35
\]

- Divisor
- Quotient
- Original polynomial
continued

b. \[ \frac{4x^4 - 7x^3 + 12x^2}{4x^3} \]

\[ \frac{4x^4 - 7x^3 + 12x^2}{4x^3} = \frac{4x^4}{4x^3} - \frac{7x^3}{4x^3} + \frac{12x^2}{4x^3} \]

\[ = x - \frac{7}{4} + \frac{3}{x} \]

**Check:**

\[ 4x^3 \left( x - \frac{7}{4} + \frac{3}{x} \right) = 4x^4 - 7x^3 + 12x^2 \]
c. \[
\frac{6a^2b^4 - 9a^3b^3 + 4a^3b^4}{a^3b^4}
\]

\[
\frac{6a^2b^4}{a^3b^4} \quad \frac{-9a^3b^3}{a^3b^4} \quad \frac{+4a^3b^4}{a^3b^4}
\]

\[
= \frac{6}{a} - \frac{9}{b} + 4
\]
Objective 2

Divide a polynomial by a polynomial of two or more terms.
EXAMPLE 2

Divide. \( \frac{2k^2 + 17k + 30}{k + 6} \)

Write the problem as if dividing whole numbers, make sure that both polynomials are written in descending powers of the variables.

\[
\begin{array}{c}
2k \\
k + 6 \overbrace{2k^2 + 17k + 30} \\
\end{array}
\]

Divide the first term of \( 2k^2 \) by the first term of \( k + 6 \).

Write the result above the division line.
continued

\[
\begin{array}{c}
\phantom{2k^2 + 17k + 30}
\end{array}
\]

\[
\begin{array}{c}
\phantom{2k^2 + 17k + 30}
\end{array}
\]

Multiply and write the result below. \(2k(k + 6)\)

Subtract. Do this mentally by changing the signs on \(2k^2 + 12k\) and adding.

Bring down 30 and continue dividing \(5k\) by \(k\).

Subtract.

You can check the result by multiplying \(k + 6\) and \(2k + 5\).
EXAMPLE 3

Divide $4x^3 + 3x - 8$ by $x + 2$.

Write the polynomials in descending order of the powers of the variables.
Add a term with 0 coefficient as a placeholder for the missing $x^2$ term.

\[
x + 2 \overline{4x^3 + 0x^2 + 3x - 8}
\]
continued

\[
\begin{array}{c}
\frac{4x^2 - 8x + 19}{x + 2} \frac{4x^3 + 0x^2 + 3x - 8}{x + 2} \\
4x^3 + 8x^2 \\
-8x^2 + 3x \\
-8x^2 - 16x \\
19x - 8 \\
19x + 38 \\
\end{array}
\]

Start with \( \frac{4x^3}{x} = 4x^2 \)

Subtract by mentally by changing the signs on \( 4x^3 + 8x^2 \) and adding.

Bring down the next term.

Next, \( \frac{-8x^2}{x} = -8x \)

Multiply then subtract.

Bring down the next term.

\( 19x/19 = 19 \)

Multiply then subtract.

The solution is: \( 4x^2 - 8x + 19 - \frac{46}{x + 2} \) and you can check the result by multiplying.
EXAMPLE 4

Divide $4m^4 - 23m^3 + 16m^2 - 4m - 1$ by $m^2 - 5m$.

Write the polynomial $m^2 - 5m$ as $m^2 - 5m + 0$.

$m^2 + 5m + 0 \overline{4m^4 - 23m^3 + 16m^2 - 4m - 1}$

Since the missing term is the last term it does not need to be written.
continued

\[
m^2 - 5m \left( \frac{4m^2 - 3m + 1}{4m^4 - 23m^3 + 16m^2 - 4m - 1} \right) \equiv \frac{4m^4 - 20m^3}{4m^4 - 20m^3} - 3m^3 + 16m^2 - 3m^3 + 15m^2 + m^2 - 4m + m^2 - 5m \equiv \frac{m - 1}{m^2 - 5m} + 4m^2 - 3m + 1 + \frac{m - 1}{m^2 - 5m}
\]
EXAMPLE 5

Divide $8x^3 + 21x^2 - 2x - 24$ by $4x + 8$.

\[ \begin{array}{c}
\text{Dividend:} \\
8x^3 + 21x^2 - 2x - 24 \\
\text{Divisor:} \\
4x + 8 \\
\end{array} \]

\[ \begin{array}{c}
\text{Quotient:} \\
2x^2 + \frac{5}{4}x - 3 \\
\text{Remainder:} \\
0 \\
\end{array} \]

The solution is:

\[ 2x^2 + \frac{5}{4}x - 3 \]
Objective 3

Divide polynomial functions.
Dividing Functions

If \( f(x) \) and \( g(x) \) define functions, then

\[
\left( \frac{f}{g} \right)(x) = \frac{f(x)}{g(x)}.
\]

The domain of the quotient function is the intersection of the domains of \( f(x) \) and \( g(x) \), excluding any values of \( x \) for which \( g(x) = 0 \).
EXAMPLE 6

For \( f(x) = 2x^2 + 17x + 30 \) and \( g(x) = 2x + 5 \), find \( \left( \frac{f}{g} \right)(x) \) and \( \left( \frac{f}{g} \right)(-1) \).

From previous Example 2, we conclude that \( \left( \frac{f}{g} \right)(x) = x + 6 \), provided the denominator \( 2x + 5 \), is not equal to zero. \( x \neq -\frac{5}{2} \)

\[
\left( \frac{f}{g} \right)(x) = \frac{f(x)}{g(x)} = \frac{2x^2 + 17x + 30}{2x + 5} = x + 6
\]

\[
\left( \frac{f}{g} \right)(-1) = -1 + 6 = 5
\]