8.1 Radical Expressions and Graphs

1. Find roots of numbers.
2. Find principal roots.
3. Graph functions defined by radical expressions.
4. Find $n$th roots of $n$th powers.
5. Use a calculator to find roots.
Objective

Find roots of numbers.
The opposite (or inverse) of squaring a number is taking its square root.

\[ \sqrt{36} = 6, \quad \text{because} \ 6^2 = 36. \]

We now extend our discussion of roots to include cube roots, fourth roots, and higher roots.

\[ \sqrt[n]{a} = b \quad \text{means} \quad b^n = a. \]
The number $a$ is the **radicand**.

$n$ is the **index** or **order**.

The expression $\sqrt[n]{a}$ is the **radical**.
EXAMPLE 1

Simplify.

a. $\sqrt[3]{27} = 3$, because $3^3 = 27$

b. $\sqrt[3]{216} = 6$, because $6^3 = 216$

c. $\sqrt[4]{256} = 4$, because $4^4 = 256$

d. $\sqrt[5]{243} = 3$, because $3^5 = 243$
continued

\[ e. \quad \sqrt[4]{\frac{16}{81}} = \frac{2}{3}, \text{ because } \left( \frac{2}{3} \right)^4 = \frac{16}{81} \]

\[ f. \quad \sqrt[3]{0.064} = 0.4, \text{ because } 0.4^3 = 0.064 \]
Objective 2

Find principal roots.
### nth Root

1. If $n$ is even and $a$ is **positive or 0**, then
   
   \[ \sqrt[n]{a} \]
   
   represents the **principal $n$th root** of $a$,

   \[ -\sqrt[n]{a} \]
   
   represents the **negative $n$th root** of $a$.

2. If $n$ is even and $a$ is **negative**, then
   
   \[ \sqrt[n]{a} \]
   
   is not a real number.

3. If $n$ is **odd**, then there is exactly one real $n$th root of $a$, written \[ \sqrt[n]{a}. \]
EXAMPLE 2

Find each root.

a. \( \sqrt{36} = 6 \)

b. \( -\sqrt{36} = -6 \)

c. \( \sqrt[4]{16} = 2 \)

d. \( -\sqrt[4]{16} = -2 \)

e. \( \sqrt[4]{-16} \)
   Not a real number.

f. \( \sqrt[5]{243} = 3 \)

g. \( \sqrt[5]{-243} = -3 \)
Objective 3

Graph functions defined by radical expressions.
Square Root Function

The domain and range of the square root function are \([0, \infty)\).
Cube Root Function

The domain and range of the cube function are \((-\infty, \infty)\).
EXAMPLE 3

Graph each function by creating a table of values. Give the domain and range.

a. \( f(x) = \sqrt{x + 2} \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>( \sqrt{-2 + 2} = 0 )</td>
</tr>
<tr>
<td>-1</td>
<td>( \sqrt{-1 + 2} = 1 )</td>
</tr>
<tr>
<td>0</td>
<td>( \sqrt{0 + 2} = 1.41 )</td>
</tr>
<tr>
<td>2</td>
<td>( \sqrt{2 + 2} = 2 )</td>
</tr>
</tbody>
</table>

Domain: \([-2, \infty)\)

Range: \([0, \infty)\)
continued

\( b. \quad f(x) = \sqrt[3]{x} - 1 \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( \sqrt[3]{0} - 1 = -1 )</td>
</tr>
<tr>
<td>1</td>
<td>( \sqrt[3]{1} - 1 = 0 )</td>
</tr>
<tr>
<td>2</td>
<td>( \sqrt[3]{2} - 1 = 1 )</td>
</tr>
<tr>
<td>-3</td>
<td>( \sqrt[3]{-3} - 1 = -1.587 )</td>
</tr>
<tr>
<td>4</td>
<td>( \sqrt[3]{4} - 1 = 1.44 )</td>
</tr>
</tbody>
</table>

Domain: \( (-\infty, \infty) \)

Range: \( (-\infty, \infty) \)
Objective 4

Find $n$th roots of $n$th powers.
For any real number \( a \), \( \sqrt{a^2} = |a| \).

That is, the principal square root of \( a^2 \) is the absolute value of \( a \).
EXAMPLE 4

Find each square root.

a. $\sqrt{15^2} = |15| = 15$

b. $\sqrt{(-12)^2} = |-12| = 12$

c. $\sqrt{y^2} = |y|

d. $\sqrt{-y^2} = |-y| = |y|$
If $n$ is an even positive integer, then $\sqrt[n]{a^n} = |a|$. 

If $n$ is an odd positive integer, then $\sqrt[n]{a^n} = a$. 

That is, use absolute value when $n$ is even; absolute value is not necessary when $n$ is odd.
EXAMPLE 5

Simplify each root.

a. \( \sqrt[4]{(-5)^4} = |-5| = 5 \)

b. \( \sqrt[5]{(-5)^5} = -5 \quad n \text{ is odd} \)

c. \( -\sqrt[6]{(-3)^6} = -|-3| = -3 \)

d. \( -\sqrt[4]{m^8} = -m^2 \quad n \text{ is even} \)
continued

d. \[ \sqrt[3]{x^{24}} = x^8 \]

e. \[ \sqrt[6]{y^{18}} = \sqrt[6]{(y^3)^6} = |y^3| \]
Objective 5

Use a calculator to find roots.
EXAMPLE 6

Use a calculator to approximate each radical to three decimal places.

a. $\sqrt{17} = 4.123$

b. $-\sqrt{362} = -19.026$

c. $\sqrt[3]{9482} = 21.166$

d. $\sqrt[4]{6825} = 9.089$
EXAMPLE 7

In electronics, the resonant frequency $f$ of a circuit may be found by the formula $f = \frac{1}{2\pi \sqrt{LC}}$ where $f$ is the cycles per second, $L$ is in henrys, and $C$ is in farads. (Henrys and farads are units of measure in electronics).

Find the resonant frequency $f$ if $L = 6 \times 10^{-5}$ and $C = 4 \times 10^{-9}$.

$$f = \frac{1}{2\pi \sqrt{LC}} \quad f = \frac{1}{2\pi \sqrt{(6 \times 10^{-5})(4 \times 10^{-9})}} \approx 324,874$$

About 325,000 cycles per second.
Rational Exponents

1. Use exponential notation for nth roots.
2. Define and use expressions of the form $a^{m/n}$.
3. Convert between radicals and rational exponents.
4. Use the rules for exponents with rational exponents.
Objective 1

Use exponential notation for $n$th roots.
If \( n^{\sqrt{a}} \) is a real number, then \( a^{1/n} = \sqrt[n]{a} \).
EXAMPLE 1

Evaluate each exponential.

a. \(32^{1/5} = \sqrt[5]{32} = 2\)

b. \(64^{1/2} = \sqrt{64} = \sqrt{64} = 8\)

c. \(-81^{1/4} = -\sqrt[4]{81} = -3\)

d. \((-81)^{1/4} = \sqrt[4]{-81}\) Is not a real number because the radicand, \(-81\), is negative and the index, 4, is even.

e. \((-64)^{1/3} = \sqrt[3]{-64} = -4\)

f. \(\left(\frac{1}{27}\right)^{1/3} = \sqrt[3]{\frac{1}{27}} = \frac{1}{3}\)
Objective 2

Define and use expressions of the form $a^{m/n}$. 
If \( m \) and \( n \) are positive integers with \( m/n \) in lowest terms, then
\[
a^{m/n} = \left(a^{1/n}\right)^m,
\]
provided that \( a^{1/n} \) is a real number. If \( a^{1/n} \) is not a real number, then \( a^{m/n} \) is not a real number.
EXAMPLE 2

Evaluate each exponential.

a. \(25^{3/2} = 25^{1/2}^3 = \sqrt{25}^3 = 5^3 = 125\)

b. \(27^{2/3} = 27^{1/3}^2 = \sqrt[3]{27}^2 = 3^2 = 9\)
c. \(-16^{3/2} = -16^{1/2}^3 = -\sqrt{16}^3 = -4^3 = -64\)

d. \(-64^{2/3} = \left[ -64^{1/3} \right]^2 = 3\sqrt{-64}^2 = -4^2 = 16\)

e. \(-32^{3/2}\) is not a real number, since \(-36^{1/2}\), or \(\sqrt{-36}\), is not a real number.
If $a^{m/n}$ is a real number, then

$$a^{-m/n} = \frac{1}{a^{m/n}} \quad (a \neq 0).$$
EXAMPLE 3

Evaluate each exponential.

a. $81^{-3/4} = \frac{1}{81^{3/4}} = \frac{1}{81^{1/4}^3} = \frac{1}{\sqrt[4]{81}^3} = \frac{1}{3^3} = \frac{1}{27}$

b. $36^{-3/2} = \frac{1}{36^{3/2}} = \frac{1}{36^{1/2}^3} = \frac{1}{\sqrt{36}^3} = \frac{1}{6^3} = \frac{1}{216}$

c. $\left(\frac{64}{25}\right)^{-3/2} = \left(\frac{25}{64}\right)^{3/2} = \left(\sqrt[2]{25}\right)^3 = \left(\frac{5}{8}\right)^3 = \frac{125}{512}$
If all indicated roots are real numbers, then

$$a^{m/n} = a^{1/n}^m = a^m^{1/n}.$$
Radical Form of $a^{m/n}$

If all indicated roots are real numbers, then

$$a^{m/n} = \sqrt[n]{a^m} = \sqrt[n]{a}^m.$$

That is, raise $a$ to the $m$th power and then take the $n$th root, or take the $n$th root of $a$ and then raise to the $m$th power.
Objective  

3

Convert between radicals and rational exponents.
EXAMPLE 4

Write each exponential as a radical. Assume that all variables, represent positive real numbers.

a. \(19^{1/2} = \sqrt[2]{19}^1 = \sqrt{19}\)

b. \(11^{3/4} = \sqrt[4]{11}^3\)

c. \(14x^{2/3} = 14 \cdot \sqrt[3]{x}^2\)

d. \(5x^{3/5} - 2x^{3/5} = 5 \cdot \sqrt[5]{x}^3 - \sqrt[5]{2x}^3\)
continued

Write each radical as an exponential.

e. \[ x^{-\frac{5}{7}} = \frac{1}{x^{\frac{5}{7}}} = \frac{1}{\sqrt[7]{x^5}} \]

f. \[ x^2 + y^2^{1/3} = \sqrt[3]{x^2 + y^2} \]

g. \[ \sqrt{37} = 37^{1/2} \]

h. \[ 4\sqrt{9^8} = 9^{8/4} = 9^2 = 81 \]

i. \[ \sqrt[8]{z^8} = z, \text{ since } z \text{ is assumed to be positive.} \]
Objective 4

Use the rules for exponents with rational exponents.
Rules for Rational Exponents

Let \( r \) and \( s \) be rational numbers. For all real numbers \( a \) and \( b \) for which the indicated expressions exist,

\[
\begin{align*}
    a^r \cdot a^s &= a^{r+s} & a^{-r} &= \frac{1}{a^r} & \frac{a^r}{b^s} &= a^{r-s} \\
    a^r^s &= a^{rs} & ab^r &= a^r b^r \\
    \left(\frac{a}{b}\right)^r &= \frac{a^r}{b^r} & \left(\frac{a}{b}\right)^{-r} &= \frac{b^r}{a^r} & a^{-r} &= \left(\frac{1}{a}\right)^r
\end{align*}
\]
EXAMPLE 5

Write with only positive exponents. Assume that all variables represent positive real numbers.

a. \(3^{1/2} \cdot 3^{1/3} = 3^{1/2 + 1/3} = 3^{3/6 + 2/6} = 3^{5/6}\)

b. \(\frac{7^{2/3}}{7^{4/3}} = 7^{2/3 - 4/3} = 7^{-2/3} = \frac{1}{7^{2/3}}\)

c. \(\left(\frac{a^{1/3} b^{2/3}}{b}\right)^6 = a^{1/3} b^{2/3 - 1} \cdot 6 = a^{1/3} b^{-1/3} \cdot 6 = a^{1/3} \cdot 6 b^{-1/3} \cdot 6\)

\[= a^{1/3} 6 b^{-1/3} 6 = a^{6/3} b^{-6/3} = a^2 b^{-2} = \frac{a^2}{b^2}\]
continued

d. \[ \left( \frac{a^3 b^{-4}}{a^{-2} b^{1/5}} \right)^{-1/2} = a^{3(-2)} b^{-4-1/5}^{-1/2} = a^{5} b^{-21/5}^{-1/2} \]

\[ = a^5^{-1/2} b^{-21/5}^{-1/2} = a^{-5/2} b^{21/10} = \frac{b^{21/10}}{a^{5/2}} \]

e. \[ r^{2/5} r^{3/5} + r^{8/5} = r^{2/5} \cdot r^{3/5} + r^{2/5} \cdot r^{8/5} \]

\[ = r^{2/5+3/5} + r^{2/5+8/5} = r^{5/5} + r^{10/5} = r + r^2 \]
EXAMPLE 6

Write all radicals as exponentials, and then apply the rules for rational exponents. Leave answers in exponential form. Assume that all variables represent positive real numbers.

a. \[ \sqrt[4]{x^3} \cdot \sqrt{x} = x^{3/4} \cdot x^{1/5} = x^{3/4 + 1/5} = x^{19/20} \]

b. \[ \frac{\sqrt{x^5}}{\sqrt[3]{x}} = \frac{x^{5/2}}{x^{1/3}} = x^{5/2 - 1/3} = x^{13/6} \]

c. \[ \sqrt[6]{x^3} = (\sqrt[6]{x})^{1/6} = x^{1/6}^{1/3} = x^{1/18} \]
Simplifying Radical Expressions

1. Use the product rule for radicals.
2. Use the quotient rule for radicals.
3. Simplify radicals.
4. Simplify products and quotients of radicals with different indexes.
5. Use the Pythagorean formula.
6. Use the distance formula.
Objective  1

Use the product rule for radicals.
Product Rule for Radicals

If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers and $n$ is a natural number, then

$$\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}.$$ 

That is, the product of two $n$th roots is the $n$th root of the product.
EXAMPLE 1

Multiply. Assume that all variables represent positive real numbers.

a. \( \sqrt{5} \cdot \sqrt{13} = \sqrt{5 \cdot 13} = \sqrt{65} \)

b. \( \sqrt{7} \cdot \sqrt{xy} = \sqrt{7xy} \)
EXAMPLE 2

Multiply. Assume that all variables represent positive real numbers.

a. \(3\sqrt{2} \cdot 3\sqrt{7} = 3\sqrt{2 \cdot 7} = 3\sqrt{14}\)

b. \(6\sqrt{8r^2} \cdot 6\sqrt{2r^3} = 6\sqrt{16r^5}\)

c. \(5\sqrt{9y^2x} \cdot 5\sqrt{8xy^2} = 5\sqrt{72y^4x^2}\)

d. \(\sqrt{7} \cdot 3\sqrt{5}\)  This expression cannot be simplified by using the product rule.
Objective 2

Use the quotient rule for radicals.
Quotient Rule for Radicals

If \( n\sqrt{a} \) and \( n\sqrt{b} \) are real numbers, \( b \neq 0 \), and \( n \) is a natural number, then

\[
\sqrt[n]{\frac{a}{b}} = \frac{n\sqrt{a}}{n\sqrt{b}}.
\]

That is, the \( n \)th root of a quotient is the quotient of the \( n \)th roots.
EXAMPLE 3

Simplify. Assume that all variables represent positive real numbers.

a. \( \sqrt{\frac{100}{81}} = \frac{10}{9} \)

b. \( \sqrt{\frac{11}{25}} = \frac{\sqrt{11}}{5} \)

c. \( \sqrt[3]{\frac{18}{125}} = \frac{\sqrt[3]{18}}{\sqrt[3]{125}} = \frac{\sqrt[3]{18}}{5} \)

d. \( \sqrt{\frac{y^8}{16}} = \frac{\sqrt{y^8}}{\sqrt{16}} = \frac{y^4}{4} \)

e. \( -\sqrt[3]{\frac{x^2}{r^{12}}} = -\frac{\sqrt[3]{x^2}}{\sqrt[3]{r^{12}}} = -\frac{\sqrt[3]{x^2}}{r^4} \)
Objective 3

Simplify radicals.
**Conditions for a Simplified Radical**

1. The radicand has no factor raised to a power greater than or equal to the index.
2. The radicand has no fractions.
3. No denominator contains a radical.
4. Exponents in the radicand and the index of the radical have no common factor (except 1).
EXAMPLE 4

Simplify.

a. \( \sqrt{32} = \sqrt{16 \cdot 2} = \sqrt{16} \cdot \sqrt{2} = 4\sqrt{2} \)

b. \( \sqrt{300} = \sqrt{100 \cdot 3} = \sqrt{100} \cdot \sqrt{3} = 10\sqrt{3} \)

c. \( \sqrt{35} \) Cannot be simplified further.
continued

d. \( \sqrt[3]{54} = \sqrt[3]{27 \cdot 2} = \sqrt[3]{27} \cdot \sqrt[3]{2} = 3\sqrt[3]{2} \)

e. \( \sqrt[4]{243} = \sqrt[4]{3^4 \cdot 3} = 3\sqrt[4]{3} \)
EXAMPLE 5

Simplify. Assume that all variables represent positive real numbers.

a. \[ \sqrt{25p^7} = \sqrt{5^2 \cdot (p^3)^2 \cdot p} = 5p^3 \sqrt{p} \]

b. \[ \sqrt[3]{72y^3x} = \sqrt[3]{36 \cdot 2 \cdot y^2 \cdot y \cdot x} = 6y \sqrt[3]{2yx} \]

c. \[ \sqrt[3]{-27y^7x^5z^6} = \sqrt[3]{-3^3 \cdot y^6 \cdot y \cdot x^3 \cdot x^2 \cdot z^6} = -3y^2xz^2 \sqrt[3]{y^2x^2} \]

d. \[ -\sqrt[4]{32a^5b^7} = -\sqrt[4]{2^4 \cdot 2 \cdot a^4 \cdot a \cdot b^4 \cdot b^3} = -2ab^4 \sqrt[4]{2ab^3} \]
EXAMPLE 6

Simplify. Assume that all variables represent positive real numbers.

a. \[ \sqrt[12]{2^3} = 2^{3^{1/12}} = 2^{1/4} = \sqrt[4]{2} \]

b. \[ \sqrt[6]{t^2} = t^{2^{1/6}} = t^{1/3} = \sqrt[3]{t} \]
If $m$ is an integer, $n$ and $k$ are natural numbers, and all indicated roots exist, then

$$\sqrt[n]{a^{km}} = \sqrt[n]{a^{m}}.$$
Objective 4

Simplify products and quotients of radicals with different indexes.
EXAMPLE 4

Simplify.

\[ \sqrt{5} \cdot \sqrt[3]{4} \]

The indexes, 2 and 3, have a least common index of 6, use rational exponents to write each radical as a sixth root.

\[ = 5^{1/2} = 5^{3/6} = \sqrt[6]{5^3} = \sqrt[6]{125} \]

\[ = 4^{1/3} = 4^{2/6} = \sqrt[6]{4^2} = \sqrt[6]{16} \]

\[ \sqrt{5} \cdot \sqrt[3]{4} = \sqrt[6]{125} \cdot \sqrt[6]{16} = \sqrt[6]{2000} \]
Objective 5

Use the Pythagorean formula.
Pythagorean Formula

If $c$ is the length of the longest side of a right triangle and $a$ and $b$ are lengths of the shorter sides, then

$$c^2 = a^2 + b^2.$$ 

The longest side is the **hypotenuse**, and the two shorter sides are the **legs**, of the triangle. The hypotenuse is the side opposite the right angle.
EXAMPLE 8

Find the length of the unknown side in each triangle.

a. \( c^2 = a^2 + b^2 \)
   
   \( c^2 = 14^2 + 8^2 \)
   
   \( c^2 = 196 + 64 \)
   
   \( c^2 = 260 \)
   
   \( c = \sqrt{260} \)
   
   \( c = \sqrt{4 \cdot 65} \)
   
   \( c = \sqrt{4} \cdot \sqrt{65} \)
   
   \( c = 2\sqrt{65} \)

   The length of the hypotenuse is \( 2\sqrt{65} \).
b. \( c^2 = a^2 + b^2 \)
\[
6^2 = 4^2 + b^2 \\
36 = 16 + b^2 \\
20 = b^2 \\
\sqrt{20} = b
\]
\[
\sqrt{4 \cdot 5} = b \\
\sqrt{4 \cdot \sqrt{5}} = b
\]
\[
2\sqrt{5} = b \quad \text{The length of the leg is } 2\sqrt{5}.
\]
Objective  6

Use the distance formula.
Distance Formula

The distance between points \((x_1, y_1)\) and \((x_2, y_2)\) is

\[ d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \]
EXAMPLE 9

Find the distance between each pair of points.

a. (2, –1) and (5, 3)

Designate which points are \((x_1, y_1)\) and \((x_2, y_2)\).

\[(x_1, y_1) = (2, -1) \text{ and } (x_2, y_2) = (5, 3)\]

\[
d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}
\]

\[
d = \sqrt{(5 - 2)^2 + (3 - (-1))^2}
\]

\[
d = \sqrt{(3)^2 + (4)^2}
\]

\[
d = \sqrt{9 + 16}
\]

\[
d = \sqrt{25} = 5
\]
continued

b. \((-3, 2)\) and \((0, -4)\)

Designate which points are \((x_1, y_1)\) and \((x_2, y_2)\).

\[(x_1, y_1) = (-3, 2)\] and \[(x_2, y_2) = (0, -4)\]

\[
d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}
\]

\[
d = \sqrt{(0 - (-3))^2 + (-4 - 2)^2}
\]

\[
d = \sqrt{3^2 + (-6)^2}
\]

\[
d = \sqrt{9 + 36}
\]

\[
d = \sqrt{45} = 3\sqrt{5}
\]
Adding and Subtracting Radical Expressions

1. Simplify radical expressions involving addition and subtraction.
Objective 1

Simplify radical expressions involving addition and subtraction.
CAUTION  Only radical expressions with the same index and the same radicand may be combined. Expressions such as $5\sqrt{3} + 2\sqrt{2}$ or $3\sqrt{3} + 2\sqrt{3}$ cannot be simplified by combining terms.
EXAMPLE 1

Add or subtract to simplify each radical expression.

a.  $3\sqrt{5} + 7\sqrt{5} = 3 + 7 \sqrt{5} = 10\sqrt{5}$

b.  $2\sqrt{11} - \sqrt{11} + 3\sqrt{44}$
    $= 2\sqrt{11} - \sqrt{11} + 3\sqrt{4} \cdot \sqrt{11}$
    $= 2\sqrt{11} - 1\sqrt{11} + 3 \cdot 2 \cdot \sqrt{11}$
    $= 2 - 1 + 6 \sqrt{11}$
    $= 7\sqrt{11}$
c. \(5\sqrt{12y} + 6\sqrt{75y}, \ y \geq 0\)

\[
= 5\sqrt{4 \cdot 3y} + 6\sqrt{25 \cdot 3y}
= 5 \cdot 2\sqrt{3y} + 6 \cdot 5\sqrt{3y}
= 10\sqrt{3y} + 30\sqrt{3y}
= 10 + 30 \sqrt{3y}
= 40\sqrt{3y}
\]

\[\text{d. } 9\sqrt{5} - 4\sqrt{10}\]

This expression can not be simplified any further.
CAUTION  Do not confuse the product rule with combining like terms. *The root of a sum does not equal the sum of the roots.* For example

\[ \sqrt{9+16} \neq \sqrt{9} + \sqrt{16} \]

since  \( \sqrt{9+16} = \sqrt{25} = 5 \), but  \( \sqrt{9} + \sqrt{16} = 3 + 4 = 7 \).
EXAMPLE 2

Add or subtract to simplify each radical expression. Assume that all variables represent positive real numbers.

a. \(-2\sqrt[4]{32} - 7\sqrt[4]{162}\) 

\[-2\sqrt[4]{16 \cdot 2} - 7\sqrt[4]{81 \cdot 2}\]

\[-2 \cdot 2\sqrt[4]{2} - 7 \cdot 3\sqrt[4]{2}\]

\[-4\sqrt[4]{2} - 21\sqrt[4]{2}\]

\[-25\sqrt[4]{2}\]
continued

b. \( \sqrt[3]{p^4 q^7} - \sqrt[3]{64pq} \)

\[
= \sqrt[3]{p^3 q^6 \cdot pq} - \sqrt[3]{64 \cdot pq} \\
= pq^2 \sqrt[3]{pq} - 4 \sqrt[3]{pq} \\
= pq^2 - 4 \sqrt[3]{pq}
\]
c. $6\sqrt[3]{16z^7} + 4\sqrt{200z^5}$

\[= 6\sqrt[3]{8z^6} \cdot 2z + 4\sqrt{100z^4} \cdot 2z\]

\[= 6\sqrt[3]{8z^6} \cdot \sqrt[3]{2z} + 4\sqrt{100z^4} \cdot \sqrt{2z}\]

\[= 6 \cdot 2z^2 \sqrt[3]{2z} + 4 \cdot 10z^2 \sqrt{2z}\]

\[= 12z^2 \sqrt[3]{2z} + 40z^2 \sqrt{2z}\]
CAUTION  Remember to write the index when working with cube roots, fourth roots, and so on.
EXAMPLE 3

Perform the indicated operations. Assume that all variables represent positive real numbers.

\[ a. \quad 2 \sqrt{\frac{32}{36}} + 2 \frac{\sqrt{27}}{\sqrt{108}} = 2 \frac{\sqrt{16 \cdot 2}}{\sqrt{36}} + 2 \frac{\sqrt{9 \cdot 3}}{\sqrt{36 \cdot 3}} \]

\[ = 2 \left( \frac{4\sqrt{2}}{6} \right) + 2 \left( \frac{3\sqrt{3}}{6\sqrt{3}} \right) \]

\[ = \frac{4\sqrt{2}}{3} + \frac{3}{3} = \frac{4\sqrt{2} + 3}{3} \]
continued

\[ b. \quad \sqrt[4]{80} y^4 + \sqrt[8]{81} y^{10} = \frac{\sqrt[4]{16 \cdot 5}}{\sqrt[4]{y^4}} + \frac{\sqrt[8]{81}}{\sqrt[8]{y^{10}}} = \frac{4\sqrt{5}}{y^2} + \frac{9}{y^5} = \frac{4y^3\sqrt{5}}{y^5} + \frac{9}{y^5} = \frac{4y^3\sqrt{5} + 9}{y^5} \]
8.5 Multiplying and Dividing Radical Expressions

1. Multiply radicals.
2. Rationalize denominators with one radical term.
3. Rationalize denominators with binomials involving radicals.
4. Write radical quotients in lowest terms.
Objective 1

Multiply radicals.
We multiply binomial expressions involving radicals by using the FOIL method from Section 5.4.
EXAMPLE 1

Multiply, using the FOIL method.

a. \(2 + \sqrt{3} \quad 1 + \sqrt{5} = 2 + 2\sqrt{5} + 1\sqrt{3} + \sqrt{15}\)

b. \(4 + \sqrt{5} \quad 4 - \sqrt{5} = 16 - 4\sqrt{5} + 4\sqrt{5} - 5\)

\[= 11 \quad \text{This is a difference of squares.}\]

c. \(\sqrt{13} - 2^2 = \sqrt{13} - 2 \quad \sqrt{13} - 2\)

\[= 13 - 2\sqrt{13} - 2\sqrt{13} + 4\]

\[= 17 - 4\sqrt{13}\]
continued

d. \[ 4 + 3\sqrt{7} \quad 4 - 3\sqrt{7} \quad = 16 - 4\sqrt{7} + 4\sqrt{7} - 3\sqrt{7^2} \]

\[ = 16 - 3\sqrt{49} \]

e. \[ \sqrt{r} + \sqrt{s} \quad \sqrt{r} - \sqrt{s} \quad = \sqrt{r}^2 - \sqrt{s}^2 \]

\[ r \geq 0 \text{ and } s \geq 0 \quad = r - s \]

Difference of squares
Objective 2

Rationalize denominators with one radical term.
EXAMPLE 2

Rationalize each denominator.

a. \( \frac{5}{\sqrt{11}} \)

Multiply the numerator and denominator by the denominator. This is in effect multiplying by 1.

\[
\frac{5}{\sqrt{11}} = \frac{5 \cdot \sqrt{11}}{\sqrt{11} \cdot \sqrt{11}} = \frac{5\sqrt{11}}{11}
\]

b. \( \frac{5\sqrt{6}}{\sqrt{5}} \)

\[
\frac{5\sqrt{6}}{\sqrt{5}} = \frac{5\sqrt{6} \cdot \sqrt{5}}{\sqrt{5} \cdot \sqrt{5}} = \frac{5\sqrt{30}}{5} = \sqrt{30}
\]
c. \[\frac{-8}{\sqrt{18}}\]

\[
\frac{-8}{\sqrt{18}} = \frac{-8 \cdot \sqrt{18}}{\sqrt{18} \cdot \sqrt{18}} = \frac{-8\sqrt{18}}{18}
\]

\[
= \frac{-8\sqrt{9 \cdot 2}}{18} = \frac{-8 \cdot 3\sqrt{2}}{18} = \frac{-24\sqrt{2}}{18} = \frac{-4\sqrt{2}}{3}
\]
EXAMPLE 3

Simplify each radical.

a. \(-\sqrt{\frac{8}{45}} = -\frac{\sqrt{8}}{\sqrt{45}}\)

   \(= -\frac{\sqrt{4 \cdot 2}}{\sqrt{9 \cdot 5}}\)

   \(= -\frac{2\sqrt{2}}{3\sqrt{5}}\)

   \(= -\frac{2\sqrt{2} \cdot \sqrt{5}}{3\sqrt{5} \cdot \sqrt{5}}\)

   Multiply by radical in denominator.

   \(= -\frac{2\sqrt{10}}{15}\)

   Product Rule
continued

\[ b. \sqrt[200k^6]{y^7}, \ y > 0 = \frac{\sqrt{200k^6}}{\sqrt{y^7}} \]

\[ = \sqrt[100 \cdot 2 \cdot (k^3)^2]{\sqrt{y^6}} \cdot y \]

\[ = 10k^3 \frac{\sqrt{2}}{y^3 \sqrt{y}} \]

\[ = \frac{10k^3 \sqrt{2} \cdot \sqrt{y}}{y^3 \sqrt{y} \cdot \sqrt{y}} = \frac{10k^3 \sqrt{2}y}{y^4} \]
EXAMPLE 4

Simplify.

a. \( \sqrt[3]{\frac{15}{32}} = \frac{\sqrt[3]{15}}{\sqrt[3]{32}} \)

\[ = \frac{\sqrt[3]{15}}{\sqrt[3]{8} \cdot \sqrt[3]{4}} \]

\[ = \frac{\sqrt[3]{15}}{2 \sqrt[3]{4}} \]

\[ = \frac{\sqrt[3]{15} \cdot \sqrt[3]{2}}{2 \sqrt[3]{4} \cdot \sqrt[3]{2}} \]

\[ = \frac{\sqrt[3]{30}}{\sqrt[3]{8}} = \frac{\sqrt[3]{30}}{4} \]

b. \( \frac{\sqrt[4]{6y}}{\sqrt[4]{w^2}}, y \geq 0, w > 0 \)

\[ = \frac{\sqrt[4]{6y}}{\sqrt[4]{w^2}} \]

\[ = \frac{\sqrt[4]{6y} \cdot \sqrt[4]{w^2}}{\sqrt[4]{w^2} \cdot \sqrt[4]{w^2}} \]

\[ = \frac{\sqrt[4]{6yw^2}}{w} \]
Objective 3

Rationalize denominators with binomials involving radicals.
To rationalize a denominator that contains a binomial expression (one that contains exactly two terms) involving radicals, such as

\[ \frac{3}{1+\sqrt{2}} \]

we must use *conjugates*. The conjugate of \(1+\sqrt{2}\) is \(1-\sqrt{2}\).
Rationalizing a Binomial Denominator

Whenever a radical expression has a sum or difference with square root radicals in the denominator, rationalize the denominator by multiplying both the numerator and denominator by the conjugate of the denominator.
EXAMPLE 5

Rationalize each denominator.

a. \[ \frac{7}{\sqrt{2} + \sqrt{13}} = \frac{7 \sqrt{2} - \sqrt{13}}{\sqrt{2} + \sqrt{13}} \]

\[ = \frac{7 \sqrt{2} - \sqrt{13}}{2 - 13} \]

\[ = \frac{7 \sqrt{2} - \sqrt{13}}{-11} = \frac{-7 \sqrt{2} - \sqrt{13}}{11} \]
b. \[
\frac{\sqrt{3} + \sqrt{5}}{\sqrt{2} - \sqrt{7}} = \frac{\sqrt{3} + \sqrt{5}}{\sqrt{2} - \sqrt{7}} \cdot \frac{\sqrt{2} + \sqrt{7}}{\sqrt{2} + \sqrt{7}}
\]

\[
= \frac{\sqrt{6} + \sqrt{21} + \sqrt{10} + \sqrt{35}}{2 - 7}
\]

\[
= \frac{\sqrt{6} + \sqrt{21} + \sqrt{10} + \sqrt{35}}{-5}
\]

\[
- \left( \sqrt{6} + \sqrt{21} + \sqrt{10} + \sqrt{35} \right)
\]

\[
= \frac{\sqrt{6} + \sqrt{21} + \sqrt{10} + \sqrt{35}}{5}
\]
continued

c. \[ \frac{2}{\sqrt{k} + \sqrt{z}}, \]

\[ k \neq z, k > 0, z > 0 \]

\[ = \frac{2 \sqrt{k} - \sqrt{z}}{\sqrt{k} + \sqrt{z}} \]

\[ - \frac{2 \sqrt{k} - \sqrt{z}}{\sqrt{k} - \sqrt{z}} \]

\[ = \frac{2 \sqrt{k} - \sqrt{z}}{k - z} \]
Objective 4

Write radical quotients in lowest terms.
EXAMPLE 6

Write each quotient in lowest terms.

a. \( \frac{24 - 36\sqrt{7}}{16} = \frac{12 \cdot 2 - 3\sqrt{7}}{16} \)

Factor the numerator and denominator.

\[ \frac{4 \cdot 3 \cdot 2 - 3\sqrt{7}}{4 \cdot 4} \]

Divide out common factors.

\[ = \frac{3 \cdot 2 - 3\sqrt{7}}{4} \]

or \( = \frac{6 - 9\sqrt{7}}{4} \)
continued

b. \( \frac{2x + \sqrt{32x^2}}{6x}, x > 0 \)  

\[ \frac{2x + 4x\sqrt{2}}{6x} \]

\[ = \frac{2x}{6x} \left( 1 + 2\sqrt{2} \right) \]

\[ = \frac{2x}{2 \cdot 3x} \left( 1 + 2\sqrt{2} \right) \]

\[ = \frac{1 + 2\sqrt{2}}{3} \]

Product rule

Factor the numerator.

Divide out common factors.
Solving Equations with Radicals

1. Solve radical equations by using the power rule.
2. Solve radical equations that require additional steps.
3. Solve radical equations with indexes greater than 2.
4. Solve radical equations by using a graphing calculator.
5. Use the power rule to solve a formula for a specified variable.
Objective 1

Solve radical equations by using the power rule.
Power Rule for Solving an Equation with Radicals

If both sides of an equation are raised to the same power, all solutions of the original equation are also solutions of the new equation.
CAUTION When the power rule is used to solve an equation, *every solution of the new equation must be checked in the original equation*.

Solutions that do not satisfy the original equation are called *extraneous solutions*; they must be discarded.
EXAMPLE 1

Solve $\sqrt{5x+1} = 4$.

$\sqrt{5x+1}^2 = 4^2$

$5x + 1 = 16$

$5x = 15$

$x = 3$

Check:

$\sqrt{5x+1} = 4$

$\sqrt{5 \cdot 3 + 1} = 4$

$\sqrt{16} = 4$

$4 = 4$

Since 3 satisfies the original equation, the solution set is $\{3\}$.
Solving an Equation with Radicals

**Step 1** Isolate the radical. Make sure that one radical term is alone on one side of the equation.

**Step 2** Apply the power rule. Raise both sides of the equation to a power that is the same as the index of the radical.

**Step 3** Solve the resulting equation; if it still contains a radical, repeat Steps 1 and 2.

**Step 4** Check all proposed solutions in the original equation.
EXAMPLE 2

Solve $\sqrt{5x + 3} + 2 = 0$.

$$\sqrt{5x + 3} = -2$$

The equation has no solution, because the square root of a real number must be nonnegative.

The solution set is $\emptyset$. 
Objective 2

Solve radical equations that require additional steps.
EXAMPLE 3

Solve \( \sqrt{5-x} = x + 1 \).

\textbf{Step 1}  The radical is alone on the left side of the equation.

\textbf{Step 2}  Square both sides. \( \sqrt{5-x}^2 = (x + 1)^2 \)

\[ 5 - x = x^2 + 2x + 1 \]

\textbf{Step 3}  The new equation is quadratic, so get 0 on one side.

\[ 0 = x^2 + 3x - 4 \]

\[ 0 = (x + 4)(x - 1) \]

\[ x + 4 = 0 \quad \text{or} \quad x - 1 = 0 \]

\[ x = -4 \quad \text{or} \quad x = 1 \]
Step 4 Check each proposed solution in the original equation.

\[
x = -4 \quad \text{or} \quad x = 1
\]

\[
\sqrt{5 - x} = x + 1
\]

\[
\sqrt{5 - (-4)} = (-4) + 1
\]

\[
\sqrt{9} = -3
\]

\[
3 \neq -3
\]

False

\[
\sqrt{5 - (1)} = (1) + 1
\]

\[
\sqrt{4} = 2
\]

\[
2 = 2
\]

True

The solution set is \{1\}. The other proposed solution, \(-4\), is extraneous.
EXAMPLE 4

Solve $\sqrt{1-2p-p^2} = p + 1$.

Step 1  The radical is alone on the left side of the equation.

Step 2  Square both sides.  $\sqrt{1-2p-p^2}^2 = (p+1)^2$

$1-2p-p^2 = p^2 + 2p + 1$

Step 3  The new equation is quadratic, so get 0 on one side.

$0 = 2p^2 + 4p$

$0 = 2p(p + 2)$

$2p = 0$  or  $p + 2 = 0$

$p = 0$  or  $p = -2$
Step 4  Check each proposed solution in the original equation.

$p = 0$  
\[
\sqrt{1 - 2p - p^2} = p + 1 \\
\sqrt{1 - 2(0) - 0^2} = 0 + 1 \\
\sqrt{1} = 1 \\
1 = 1
\]

or  
\[
p = -2 \\
\sqrt{1 - 2p - p^2} = p + 1 \\
\sqrt{1 - 2(-2) - (-2)^2} = -2 + 1 \\
\sqrt{1} \neq -1 \\
\sqrt{1} = 1
\]

True  
False

The solution set of the original equation is \{0\}.
EXAMPLE 5

Solve $\sqrt{2x+3} + \sqrt{x+1} = 1$.

\[
\sqrt{2x+3} = 1 - \sqrt{x+1}
\]

Square both sides.

\[
\sqrt{2x+3}^2 = (1 - \sqrt{x+1})^2
\]

1. Square both sides.

\[
2x + 3 = 1 - 2\sqrt{x+1} + (x+1)
\]

Isolate the remaining radical.

\[
x + 1 = -2\sqrt{x+1}
\]

Square both sides.

\[
x + 1 = (-2\sqrt{x+1})^2
\]

Apply the exponent rule $(ab)^2 = a^2b^2$.
continued

\[ x^2 + 2x + 1 = 4(x + 1) \]
\[ x^2 + 2x + 1 = 4x + 4 \]
\[ x^2 - 2x - 3 = 0 \]
\[ (x - 3)(x + 1) = 0 \]
\[ x - 3 = 0 \quad \text{or} \quad x + 1 = 0 \]
\[ x = 3 \quad \text{or} \quad x = -1 \]

Check: \[ x = 3 \]
\[ \sqrt{2(3) + 3} + \sqrt{3+1} = 1 \]
\[ \sqrt{6 + 3} + \sqrt{4} = 1 \]
\[ \sqrt{9} + \sqrt{4} = 1 \]
\[ 3 + 2 = 1 \]
False \[ 5 \neq 1 \]

Check: \[ x = -1 \]
\[ \sqrt{2(-1) + 3} + \sqrt{-1+1} = 1 \]
\[ \sqrt{1} + \sqrt{0} = 1 \]
\[ 1 = 1 \]
True

The solution set is \{ -1 \}. 

Objective 3

Solve radical equations with indexes greater than 2.
EXAMPLE 6

Solve $\sqrt[3]{2x + 7} = \sqrt[3]{3x - 2}$.

Cube both sides.

$\sqrt[3]{2x + 7}^3 = \sqrt[3]{3x - 2}^3$

$2x + 7 = 3x - 2$

$x = 9$

Check:

$\sqrt[3]{2(9) + 7} = \sqrt[3]{3(9) - 2}$

$\sqrt[3]{18 + 7} = \sqrt[3]{27 - 2}$

$\sqrt[3]{25} = \sqrt[3]{25}$

The solution set is $\{9\}$. True
Objective 4

Solve radical equations by using a graphing calculator.
EXAMPLE

Solve \( \sqrt{2x + 3} + \sqrt{x + 1} = 1 \) using a graphing calculator.

\[
\sqrt{2x + 3} + \sqrt{x + 1} - 1 = 0
\]
EXAMPLE

Solve $\sqrt{1-2p-p^2} = p + 1$ using a graphing calculator.

$\sqrt{1-2p-p^2} - p - 1 = 0$
Objective 5

Use the power rule to solve a formula for a specified variable.
EXAMPLE 7

Solve the formula for \( L \).

\[ Z = \sqrt{\frac{L}{C}} \]

\[ (Z)^2 = \left( \sqrt{\frac{L}{C}} \right)^2 \]

\[ Z^2 = \frac{L}{C} \]

\[ CZ^2 = L \]
8.7 Complex Numbers

1. Simplify numbers of the form $\sqrt{-b}$, where $b > 0$.
2. Recognize complex numbers.
3. Add and subtract complex numbers.
5. Divide complex numbers.
6. Find powers of $i$. 
Objective 1

Simplify numbers of the form $\sqrt{-b}$, where $b > 0$. 
The imaginary unit $i$ is defined as

$$i = \sqrt{-1}, \quad \text{where} \quad i^2 = -1.$$ 

That is, $i$ is the principal square root of $-1$. 
For any positive real number \( b \), \( \sqrt{-b} = i\sqrt{b} \).
EXAMPLE 1

Write each number as a product of a real number and $i$.

a. $\sqrt{-25} = i\sqrt{25} = 5i$

b. $-\sqrt{-81} = -i\sqrt{81} = -9i$

c. $\sqrt{-7} = i\sqrt{7}$

d. $\sqrt{-44} = i\sqrt{44} = i\sqrt{4 \cdot 11} = 2i\sqrt{11}$
CAUTION  It is easy to mistake $\sqrt{2i}$ for $\sqrt{2i}$, with the $i$ under the radical.

For this reason, we usually write $\sqrt{2i}$ as $i\sqrt{2}$ as in the definition of $\sqrt{-b}$. 
EXAMPLE 2

Multiply.

a. \( \sqrt{-6} \cdot \sqrt{-5} = i \sqrt{6} \cdot i \sqrt{5} \)

\[
= i^2 \sqrt{6 \cdot 5}\\
= (-1) \sqrt{30} = -\sqrt{30}
\]

b. \( \sqrt{-8} \cdot \sqrt{-6} = i \sqrt{8} \cdot i \sqrt{6} \)

\[
= i^2 \sqrt{8 \cdot 6}\\
= i^2 \sqrt{48}\\
= i^2 \sqrt{16 \cdot 3} = -4 \sqrt{3}
\]

c. \( \sqrt{-5} \cdot \sqrt{7} \)

\[
= i \sqrt{5} \cdot \sqrt{7}\\
= i \sqrt{35}
\]
EXAMPLE 3

Divide.

a. \[\frac{\sqrt{-80}}{\sqrt{-5}} = \frac{i\sqrt{80}}{i\sqrt{5}}\]
   \[= \frac{\sqrt{80}}{\sqrt{5}}\]
   \[= \sqrt{16}\]
   \[= 4\]

b. \[\frac{\sqrt{-40}}{\sqrt{10}} = \frac{i\sqrt{40}}{\sqrt{10}}\]
   \[= i\sqrt{4}\]
   \[= 2i\]
Objective 2

Recognize complex numbers.
If $a$ and $b$ are real numbers, then any number of the form $a + bi$ is called a complex number. In the complex number $a + bi$, the number $a$ is called the real part and $b$ is called the imaginary part.
For a complex number $a + bi$, if $b = 0$, then $a + bi = a$, which is a real number.

Thus, the set of real numbers is a subset of the set of complex numbers.

If $a = 0$ and $b \neq 0$, the complex number is said to be a pure imaginary number.

For example, $3i$ is a pure imaginary number. A number such as $7 + 2i$ is a nonreal complex number.

A complex number written in the form $a + bi$ is in standard form.
The relationships among the various sets of numbers.
Objective 3

Add and subtract complex numbers.
EXAMPLE 4

Add.

a. \((-1 - 8i) + (9 - 3i) = (-1 + 9) + (-8 - 3)i\)
   \[= 8 - 11i\]

b. \((-3 + 2i) + (1 - 3i) + (-7 - 5i)\)
   \[= [-3 + 1 + (-7)] + [2 + (-3) + (-5)]i\]
   \[= -9 - 6i\]
EXAMPLE 5

Subtract.

a. \(( -1 + 2i ) - ( 4 + i )\)  
   \[ \begin{align*}
   & = ( -1 - 4 ) + ( 2 - 1)i \\
   & = -5 + i
   \end{align*} \]

b. \(( 8 - 5i ) - ( 12 - 3i )\)  
   \[ \begin{align*}
   & = ( 8 - 12 ) + [ -5 - ( -3 )]i \\
   & = ( 8 - 12 ) + ( -5 + 3)i \\
   & = -4 - 2i
   \end{align*} \]
c. \((-10 + 6i) - (-10 + 10i)\)

\[\begin{align*}
&= [-10 - (-10)] + (6 - 10)i \\
&= 0 - 4i \\
&= -4i
\end{align*}\]
Objective

Multiply complex numbers.
EXAMPLE 6

Multiply.

a. \(6i(4 + 3i) = 6i(4) + 6i(3i)\)

\[= 24i + 18i^2\]

\[= 24i + 18(-1)\]

\[= -18 + 24i\]
b. \((6 - 4i)(2 + 4i) = 6(2) + 6(4i) + (-4i)(2) + (-4i)(4i)\)

\[
= 12 + 24i - 8i - 16i^2
\]

\[
= 12 + 16i - 16(-1)
\]

\[
= 12 + 16i + 16
\]

\[
= 28 + 16i
\]
c. \((3 + 2i)(3 + 4i)\) = \[3(3) + 3(4i) + (2i)(3) + (2i)(4i)\]

\[= 9 + 12i + 6i + 8i^2\]

\[= 9 + 18i + 8(-1)\]

\[= 9 + 18i - 8\]

\[= 1 + 18i\]
The product of a complex number and its conjugate is always a real number.

\[(a + bi)(a - bi) = a^2 - b^2(-1)\]
\[= a^2 + b^2\]
Objective 5

Divide complex numbers.
EXAMPLE 7

Find each quotient.

\[ \frac{6 + 2i}{4 - 3i} = \frac{(6 + 2i)(4 + 3i)}{(4 - 3i)(4 + 3i)} \]

\[ = \frac{24 + 18i + 8i + 6i^2}{4^2 + 3^2} \]

\[ = \frac{24 + 26i + 6(-1)}{16 + 9} \]

\[ = \frac{18 + 26i}{25} \text{ or } \frac{18}{25} + \frac{26i}{25} \]

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continued

b. \[
\frac{5 - i}{i} = \frac{(5 - i)(-i)}{i(-i)}
\]

\[
= \frac{-5i + i^2}{-i^2}
\]

\[
= \frac{-5i + (-1)}{-(-1)}
\]

\[
= \frac{-5i - 1}{1} = -1 - 5i
\]
Objective  6

Find powers of $i$. 
Because $i^2 = -1$, we can find larger powers of $i$, as shown below.

\[ i^3 = i \cdot i^2 = i \cdot (-1) = -i \]
\[ i^4 = i^2 \cdot i^2 = (-1) \cdot (-1) = 1 \]
\[ i^5 = i \cdot i^4 = i \cdot 1 = i \]
\[ i^6 = i^2 \cdot i^4 = (-1) \cdot (1) = -1 \]
\[ i^7 = i^3 \cdot i^4 = (-i) \cdot (1) = -i \]
\[ i^8 = i^4 \cdot i^4 = 1 \cdot 1 = 1 \]
EXAMPLE 8

Find each power of $i$.

a. $i^{28} = i^4 \cdot 7 = 1^7 = 1$

b. $i^{19} = i^{16} \cdot i^3 = i^4 \cdot 4 \cdot i^3 = 1^4 \cdot (-i) = -i$

c. $i^{-9} = \frac{1}{i^9} = \frac{1}{i^8 \cdot i} = \frac{1}{i^4 \cdot 2 \cdot i} = \frac{1}{1^2 \cdot i} = \frac{1}{i}$

$= \frac{1(-i)}{i \cdot (-i)} = \frac{-i}{-i^2} = \frac{-i}{-(-1)} = \frac{-i}{1} = -i$
d. \( i^{-22} = \frac{1}{i^{22}} = \frac{1}{i^{20} \cdot i^2} = \frac{1}{i^4 \cdot 5 \cdot (-1)} = \frac{1}{1^5 \cdot (-1)} = \frac{1}{-1} = -1 \)