The Square Root Property and Completing the Square

1. Review the zero-factor property.
2. Learn the square root property.
3. Solve quadratic equations of the form $(ax + b)^2 = c$ by using the square root property.
4. Solve quadratic equations by completing the square.
5. Solve quadratic equations with solutions that are not real numbers.
Quadratic Equation

An equation that can be written in the form

\[ ax^2 + bx + c = 0, \]

where \( a, b, \) and \( c \) are real numbers, with \( a \neq 0 \), is a **quadratic equation**. The given form is called **standard form**.
Objective 1

Review the zero-factor property.
Zero-Factor Property

If two numbers have a product of 0, then at least one of the numbers must be 0. That is, if \( ab = 0 \), then \( a = 0 \) or \( b = 0 \).
Objective 2

Learn the square root property.
### Square Root Property

If \( x \) and \( k \) are complex numbers and \( x^2 = k \), then

\[
x = \sqrt{k} \quad \text{or} \quad x = -\sqrt{k}.
\]

**CAUTION**  Remember that if \( k \neq 0 \), using the square root property always produces two square roots, one positive and one negative.
EXAMPLE 1

Solve each equation.

a. \( m^2 = 64 \)  
   By the square root property, \( m = 8 \) or \( m = -8 \)  
   The solution set is \( \{-8, 8\} \)

b. \( 3x^2 - 54 = 0 \)
   \[ 3x^2 = 54 \]
   \[ x^2 = 18 \]
   By the square root property,
   \[ x = \sqrt{18} \quad \text{or} \quad x = -\sqrt{18} \]
   \[ x = 3\sqrt{2} \quad \text{or} \quad x = -3\sqrt{2} \]
   Check:
   \[ x = \pm 3\sqrt{2}: \quad 3\sqrt{2} \cdot -3\sqrt{2} = 54 \]
   True
   Solution set: \( 3\sqrt{2}, -3\sqrt{2} \).
EXAMPLE 2

An expert marksman can hold a silver dollar at forehead level, drop it, draw his gun, and shoot the coin as it passes waist level. If the coin falls about 4 ft, use the formula \( d = 16t^2 \) to find the time that elapses between the dropping of the coin and the shot.

\[
d = 16t^2 \\
4 = 16t^2 \\
\frac{1}{4} = t^2
\]

By the square root property,

\[
t = \frac{1}{2} \quad \text{or} \quad t = -\frac{1}{2}
\]

Since time cannot be negative, we discard the negative solution. Therefore, 0.5 sec elapses between the dropping of the coin and the shot.
Objective 3

Solve quadratic equations of the form \((ax + b)^2 = c\) by using the square root property.
EXAMPLE 3

Solve \((3x + 1)^2 = 2\).

\[
3x + 1 = \sqrt{2} \quad \text{or} \quad 3x + 1 = -\sqrt{2}
\]

\[
3x = -1 + \sqrt{2} \quad \text{or} \quad 3x = -1 - \sqrt{2}
\]

\[
x = \frac{-1 + \sqrt{2}}{3} \quad \text{or} \quad x = \frac{-1 - \sqrt{2}}{3}
\]
continued

We show a check for the first solution. The check for the other solution is similar.

Check: \((3x + 1)^2 = 2\)

\[
\left[ 3 \left( \frac{-1 + \sqrt{2}}{3} \right) + 1 \right]^2 = 2
\]

\(-1 + \sqrt{2} + 1 = 2\)

\(\sqrt{2} = 2\)

\(2 = 2\)

Let \(x = \frac{-1 + \sqrt{2}}{3}\).

The solution set is \(\left\{ \frac{-1 + \sqrt{2}}{3}, \frac{-1 - \sqrt{2}}{3} \right\}\).
Objective 4

Solve quadratic equations by completing the square.
EXAMPLE 4

Solve \( x^2 - 2x - 10 = 0 \).

\[ x^2 - 2x = 10 \]

Completing the square

\[
\left[ \frac{1}{2} \ -2 \right]^2 = -1 \ = 1.
\]

Add 1 to each side.

\[ x^2 - 2x + 1 = 10 + 1 \]

\[ (x - 1)^2 = 11 \]
Use the square root property.

\[ x - 1 = \sqrt{11} \quad \text{or} \quad x - 1 = -\sqrt{11} \]

\[ x = 1 + \sqrt{11} \quad \text{or} \quad x = 1 - \sqrt{11} \]

**Check:** \( x = 1 + \sqrt{11} : \)

\[ (1 + \sqrt{11})^2 - 2 \cdot 1 + \sqrt{11} - 10 = 0 \]

\[ 12 + 2\sqrt{11} - 2 - 2\sqrt{11} - 10 = 0 \]

True

The solution set is \( 1 + \sqrt{11}, 1 - \sqrt{11} \).
Completing the Square

To solve $ax^2 + bx + c = 0 \ (a \neq 0)$ by completing the square, use these steps.

**Step 1** Be sure the second-degree (squared) term has coefficient 1. If the coefficient of the squared term is one, proceed to Step 2. If the coefficient of the squared term is not 1 but some other nonzero number $a$, divide each side of the equation by $a$.

**Step 2** Write the equation incorrect form so that terms with variables are on one side of the equals sign and the constant is on the other side.

**Step 3** Square half the coefficient of the first-degree (linear) term.
Completing the Square (continued)

**Step 4** Add the square to each side.

**Step 5** Factor the perfect square trinomial. One side should now be a perfect square trinomial. Factor it as the square of a binomial. Simplify the other side.

**Step 6** Solve the equation. Apply the square root property to complete the solution.

**NOTE** Steps 1 and 2 can be done in either order. With some equations, it is more convenient to do Step 2 first.
EXAMPLE 5

Solve \( x^2 + 3x - 1 = 0 \).

\[ x^2 + 3x = 1 \]

Completing the square.

Add the square to each side.

\[ \left( \frac{1}{2} x + 3 \right)^2 = \left( \frac{3}{2} \right)^2 = \frac{9}{4} \]

\[ x^2 + 3x + \frac{9}{4} = 1 + \frac{9}{4} \]

\[ \left( x + \frac{3}{2} \right)^2 = \frac{13}{4} \]
Use the square root property.

\[ x + \frac{3}{2} = \sqrt{\frac{13}{4}} \quad \text{or} \quad x + \frac{3}{2} = -\sqrt{\frac{13}{4}} \]

\[ x + \frac{3}{2} = \frac{\sqrt{13}}{2} \quad \text{or} \quad x + \frac{3}{2} = -\frac{\sqrt{13}}{2} \]

\[ x = \frac{-3 + \sqrt{13}}{2} \quad \text{or} \quad x = \frac{-3 - \sqrt{13}}{2} \]

Check that the solution set is \( \left\{ \frac{-3 + \sqrt{13}}{2}, \frac{-3 - \sqrt{13}}{2} \right\} \).
EXAMPLE 6

Solve $3x^2 + 6x - 2 = 0$.

$$3x^2 + 6x = 2$$

$$x^2 + 2x = \frac{2}{3}$$

Completing the square.

$$\left[ \frac{1}{2} \ 2 \right]^2 = 1^2 = 1$$

$$x^2 + 2x + 1 = \frac{2}{3} + 1$$

$$x + 1^2 = \frac{5}{3}$$
continued

Use the square root property.

\[ x + 1 = \sqrt{\frac{5}{3}} \quad \text{or} \quad x + 1 \frac{3}{2} = -\sqrt{\frac{5}{3}} \]

\[ x = -1 + \sqrt{\frac{5}{3}} \quad \text{or} \quad x = -1 - \sqrt{\frac{5}{3}} \]

\[ x = -1 + \frac{\sqrt{15}}{3} \quad \text{or} \quad x = -1 - \frac{\sqrt{15}}{3} \]

\[ x = \frac{-3 + \sqrt{15}}{3} \quad \text{or} \quad x = \frac{-3 - \sqrt{15}}{3} \]
Check that the solution set is

\[
\left\{ \frac{-3 + \sqrt{15}}{3}, \frac{-3 - \sqrt{15}}{3} \right\}.
\]
Objective 5

Solve quadratic equations with solutions that are not real numbers.
EXAMPLE 7

Solve each equation.

a. \( x^2 = -17 \)

\[
x = \sqrt{-17} \quad \text{or} \quad x = -\sqrt{-17}
\]

\[
x = i\sqrt{17} \quad \text{or} \quad x = -i\sqrt{17}
\]

The solution set is

\[i\sqrt{17} , -i\sqrt{17}\]
b. \( x + 5^2 = -100 \)

\[ x + 5 = \sqrt{-100} \quad \text{or} \quad x + 5 = -\sqrt{-100} \]

\[ x + 5 = 10i \quad \text{or} \quad x + 5 = -10i \]

\[ x = -5 + 10i \quad \text{or} \quad x = -5 - 10i \]

The solution set is \(-5 + 10i, -5 - 10i\).
c. \( 5x^2 - 15x + 12 = 0 \)

\[ 5x^2 - 15x = -12 \]

\[ x^2 - 3x = -\frac{12}{5} \]

Complete the square.

\[ \left[ \frac{1}{2} - 3 \right]^2 = \left( -\frac{3}{2} \right)^2 = \frac{9}{4} \]

\[ x^2 - 3x + \frac{9}{4} = -\frac{12}{5} + \frac{9}{4} \]

\[ \left( x - \frac{3}{2} \right)^2 = \frac{3}{20} \]
continued

\[ x - \frac{3}{2} = \sqrt{-\frac{3}{20}} \quad \text{or} \quad x - \frac{3}{2} = -\sqrt{-\frac{3}{20}} \]

\[ x - \frac{3}{2} = \frac{i\sqrt{3}}{\sqrt{20}} \cdot \frac{\sqrt{5}}{\sqrt{5}} \quad \text{or} \quad x - \frac{3}{2} = \frac{-i\sqrt{3}}{\sqrt{20}} \cdot \frac{\sqrt{5}}{\sqrt{5}} \]

\[ x - \frac{3}{2} = \frac{i\sqrt{15}}{10} \quad \text{or} \quad x - \frac{3}{2} = \frac{-i\sqrt{15}}{10} \]

\[ x = \frac{3}{2} + \frac{i\sqrt{15}}{10} \quad \text{or} \quad x = \frac{3}{2} - \frac{i\sqrt{15}}{10} \]
Solution set:

\[ \left\{ \frac{3}{2} + \frac{\sqrt{15}}{10} \, i, \quad \frac{3}{2} - \frac{\sqrt{15}}{10} \, i \right\} \]
Derive the quadratic formula.
Solve quadratic equations by using the quadratic formula.
Use the discriminant to determine the number and type of solutions.
Objective 1

Derive the quadratic formula.
Solve \( ax^2 + bx + c = 0 \) by completing the square (assuming \( a > 0 \)).

\[
ax^2 + bx + c = 0
\]

\[
x^2 + \frac{b}{a}x + \frac{c}{a} = 0
\]

\[
x^2 + \frac{b}{a}x = -\frac{c}{a}
\]

\[
\left[ \frac{1}{2} \left( \frac{b}{a} \right) \right]^2 = \left( \frac{b}{2a} \right)^2 = \frac{b^2}{4a^2}
\]

\[
x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = -\frac{c}{a} + \frac{b^2}{4a^2}
\]

\[
\left( x + \frac{b}{2a} \right)^2 = \frac{b^2}{4a^2} + \frac{-c}{a}
\]

\[
\left( x + \frac{b}{2a} \right)^2 = \frac{b^2}{4a^2} + \frac{-4ac}{4a^2}
\]

\[
\left( x + \frac{b}{2a} \right)^2 = \frac{b^2 - 4ac}{4a^2}
\]

\[
x + \frac{b}{2a} = \sqrt{\frac{b^2 - 4ac}{4a^2}}
\]

or \[
x + \frac{b}{2a} = -\sqrt{\frac{b^2 - 4ac}{4a^2}}
\]
\[ x + \frac{b}{2a} = \sqrt{\frac{b^2 - 4ac}{4a^2}} \quad \text{or} \quad x + \frac{b}{2a} = -\sqrt{\frac{b^2 - 4ac}{4a^2}} \]

\[ x + \frac{b}{2a} = \frac{\sqrt{b^2 - 4ac}}{2a} \quad \text{or} \quad x + \frac{b}{2a} = \frac{-\sqrt{b^2 - 4ac}}{2a} \]

\[ x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{or} \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \]
### Quadratic Formula

The solutions of the equation $ax^2 + bx + c = 0$ ($a \neq 0$) are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$
Objective 2

Solve quadratic equations by using the quadratic formula.
EXAMPLE 1

Solve $4x^2 - 11x - 3 = 0$.

$a = 4$, $b = \ -11$ and $c = \ -3$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-11) \pm \sqrt{(-11)^2 - 4(4)(-3)}}{2(4)}$$

$$x = \frac{11 \pm \sqrt{121 + 48}}{8}$$

$$x = \frac{11 \pm \sqrt{169}}{8}$$

$$x = \frac{11 \pm 13}{8}$$

$$x = \frac{11 \pm 13}{8}$$

$$x = \frac{24}{8} = 3$$

$$x = \frac{11 - 13}{8}$$

$$x = \frac{11 - 13}{8}$$

$$x = \frac{-2}{8} = -\frac{1}{4}$$

The solution set is \{-1/4, 3\}. 
EXAMPLE 2

Solve $2x^2 + 19 = 14x$. 

$2x^2 - 14x + 19 = 0$

$a = 2$, $b = -14$ and $c = 19$

$x = \frac{-b \pm \sqrt{(b)^2 - 4(a)(c)}}{2(a)}$

$x = \frac{-(-14) \pm \sqrt{(-14)^2 - 4(2)(19)}}{2(2)}$

$x = \frac{14 \pm \sqrt{196 - 152}}{4}$

$x = \frac{14 \pm \sqrt{44}}{4}$

$x = \frac{14 \pm \sqrt{4 \cdot 11}}{4}$

$x = \frac{14 + 2\sqrt{11}}{4} = \frac{7 + \sqrt{11}}{2}$

$x = \frac{14 - 2\sqrt{11}}{4} = \frac{7 - \sqrt{11}}{2}$

The solution set is $\left\{ \frac{7 \pm \sqrt{11}}{2} \right\}$. 
EXAMPLE 3

Solve \((x + 5)(x + 1) = 10x\).

\[ x^2 + 6x + 5 = 10x \]
\[ x^2 - 4x + 5 = 0 \]

\(a = 1, \ b = -4\) and \(c = 5\)

\[ x = \frac{-b \pm \sqrt{(b)^2 - 4(a)(c)}}{2(a)} \]

\[ x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(5)}}{2(1)} \]

\[ x = \frac{4 \pm \sqrt{16 - 20}}{4} \]
\[ x = \frac{4 \pm \sqrt{-4}}{2} \]
\[ x = \frac{4 \pm 2i}{2} \]

The solution set is \(2 \pm i\).
Objective 3

Use the discriminant to determine the number and type of solutions.
The discriminant of $ax^2 + bx + c = 0$ is $b^2 - 4ac$. If $a$, $b$, and $c$ are integers, then the number and type of solutions are determined as follows.

<table>
<thead>
<tr>
<th>Discriminant</th>
<th>Number and Type of Solutions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Positive, and the square of an integer</td>
<td>Two rational solutions</td>
</tr>
<tr>
<td>Positive, but not the square of an integer</td>
<td>Two irrational solutions</td>
</tr>
<tr>
<td>Zero</td>
<td>One rational solution</td>
</tr>
<tr>
<td>Negative</td>
<td>Two nonreal complex solutions</td>
</tr>
</tbody>
</table>
EXAMPLE 4

Find each discriminant. Use it to predict the number and type of solutions for each equation. Tell whether the equation can be solved by factoring or whether the quadratic formula should be used.

a. \(10x^2 - x - 2 = 0\)

\[a = 10, \quad b = -1, \quad c = -2\]

\[b^2 - 4ac = (-1)^2 - 4(10)(-2)\]

\[= 1 + 80\]

\[= 81\]

There will be two rational solutions, and the equation can be solved by factoring.
continued

b. \(3x^2 - x = 7\)
\[3x^2 - x - 7 = 0\]
\[a = 3, \ b = -1, \ c = -7\]
\[b^2 - 4ac = (-1)^2 - 4(3)(-7)\]
\[= 1 + 84\]
\[= 85\]

There will be two irrational solutions. Solve by using the quadratic formula.

c. \(16x^2 + 25 = 40x\)
\[16x^2 - 40x + 25 = 0\]
\[a = 16, \ b = -40, \ c = 25\]
\[b^2 - 4ac = (-40)^2 - 4(16)(25)\]
\[= 1600 - 1600\]
\[= 0\]

There will be one rational solution. Solve by factoring.
EXAMPLE 5

Find $k$ so that the equation will have exactly one rational solution.

$$x^2 - kx + 64 = 0$$

$$b^2 - 4ac = (-k)^2 - 4(1)(64)$$

$$= k^2 - 256$$

$$k^2 - 256 = 0$$

$$k^2 = 256$$

$$k = 16 \quad \text{or} \quad k = -16$$

There will be only one rational solution if $k = 16$ or $k = -16$. 
9.3 Equations Quadratic in Form

1. Solve an equation with fractions by writing it in quadratic form.
2. Use quadratic equations to solve applied problems.
3. Solve an equation with radicals by writing it in quadratic form.
4. Solve an equation that is quadratic in form by substitution.
METHODS FOR SOLVING QUADRATIC EQUATIONS

<table>
<thead>
<tr>
<th>Method</th>
<th>Advantages</th>
<th>Disadvantages</th>
</tr>
</thead>
<tbody>
<tr>
<td>Factoring</td>
<td>This is usually the fastest method.</td>
<td>Not all polynomials are factorable; some factorable polynomials are difficult to factor.</td>
</tr>
<tr>
<td>Square root property</td>
<td>This is the simplest method for solving equations of the form ((ax + b)^2 = c).</td>
<td>Few equations are given in this form.</td>
</tr>
<tr>
<td>Completing the square</td>
<td>This method can always be used, although most people prefer the quadratic formula.</td>
<td>It requires more steps than other methods.</td>
</tr>
<tr>
<td>Quadratic formula</td>
<td>This method can always be used.</td>
<td>It is more difficult than factoring because of the square root, although calculators can simplify its use.</td>
</tr>
</tbody>
</table>
Objective 1

Solve an equation with fractions by writing it in quadratic form.
EXAMPLE 1

Solve \( \frac{4}{x-1} + 9 = -\frac{7}{x} \).

Multiply by the LCD, \( x(x - 1) \).

\[
x(x - 1) \left( \frac{4}{x-1} + 9 \right) = x(x - 1) \left( -\frac{7}{x} \right)
\]

\[
4x + 9x - 1 = -7x - 1
\]

\[
4x + 9x^2 - 9x = -7x + 7
\]

\[
9x^2 + 2x - 7 = 0
\]
continued

\[ 9x^2 + 2x - 7 = 0 \]

\[ 9x - 7 \quad x + 1 \quad = 0 \]

\[ 9x - 7 = 0 \quad \text{or} \quad x + 1 = 0 \]

\[ x = \frac{7}{9} \quad \text{or} \quad x = -1 \]

The solution set is \( \left\{ -1, \frac{7}{9} \right\} \).
Objective 2

Use quadratic equations to solve applied problems.
EXAMPLE 2

In 1 ¾ hr Khe rows his boat 5 mi upriver and comes back. The speed of the current is 3 mph. How fast does Khe row?

**Step 1**  **Read** the problem carefully.

**Step 2**  **Assign the variable.** Let \( x \) = the speed Khe can row.  Make a table. Use \( t = \frac{d}{r} \).

<table>
<thead>
<tr>
<th></th>
<th>( d )</th>
<th>( r )</th>
<th>( t )</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Upstream</strong></td>
<td>5</td>
<td>( x - 3 )</td>
<td>( \frac{5}{x - 3} )</td>
</tr>
<tr>
<td><strong>Downstream</strong></td>
<td>5</td>
<td>( x + 3 )</td>
<td>( \frac{5}{x + 3} )</td>
</tr>
</tbody>
</table>

Times in hours.
continued

**Step 3**  **Write an equation.** The time going upriver added to the time going downriver is $1\frac{3}{4}$ or $\frac{7}{4}$ hr.

$$\frac{5}{x-3} + \frac{5}{x+3} = \frac{7}{4}$$

**Step 4**  **Solve** the equation. Multiply each side by the LCD, $4(x - 3)(x + 3)$.

$$4 \cdot \frac{x-3}{x-3} \cdot \frac{5}{x-3} + 4 \cdot \frac{x+3}{x+3} \cdot \frac{5}{x+3} = 4 \cdot \frac{7}{4} \cdot \frac{x-3}{x-3} \cdot \frac{x+3}{x+3}$$

$$20 \cdot \frac{x+3}{x+3} + 20 \cdot \frac{x-3}{x-3} = 7 \cdot \frac{x-3}{x-3} \cdot \frac{x+3}{x+3}$$
\[ 20x + 60 + 20x - 60 = 7 \quad x^2 - 9 \]
\[ 40x = 7x^2 - 63 \]
\[ 0 = 7x^2 - 40x - 63 \]
\[ 0 = 7x + 9 \quad x - 7 \]
\[ 7x + 9 \quad \text{or} \quad x - 7 \]
\[ x = -\frac{9}{7} \quad \text{or} \quad x = 7 \]

**Step 5** State the answer. The speed cannot be negative, so Khe rows at the speed of 7mph.

**Step 6** Check that this value satisfies the original problem.
EXAMPLE 3

Two chefs are preparing a banquet. One chef could prepare the banquet in 2 hr less time than the other. Together, they complete the job in 5 hr. How long would it take the faster chef working alone?

**Step 1**  Read the problem carefully.

**Step 2**  Assign the variable. Let $x =$ the slow chef’s time alone. Then $x - 2 =$ the fast chef’s time alone.

<table>
<thead>
<tr>
<th></th>
<th>Rate</th>
<th>Time working Together</th>
<th>Fractional Part of the Job Done</th>
</tr>
</thead>
<tbody>
<tr>
<td>Slow</td>
<td>$\frac{1}{x}$</td>
<td>5</td>
<td>$\frac{5}{x}$</td>
</tr>
<tr>
<td>Fast</td>
<td>$\frac{1}{x-2}$</td>
<td>5</td>
<td>$\frac{5}{x-2}$</td>
</tr>
</tbody>
</table>
continued

Step 3 Write an equation. Since together they
complete 1 job, \( \frac{5}{x} + \frac{5}{x-2} = 1 \).

Step 4 Solve the equation. Multiply each side by
the LCD, \( x(x-2) \).

\[
x \cdot \frac{5}{x} + x \cdot \frac{5}{x-2} = x \cdot (x-2) \cdot 1
\]

\[
5 \cdot (x-2) + 5x = x \cdot x - 2
\]

\[
5x - 10 + 5x = x^2 - 2x
\]

\[
0 = x^2 - 12x + 10
\]
Here \( a = 1, \ b = -12, \) and \( c = 10. \)

\[
x = \frac{-(-12) \pm \sqrt{(-12)^2 - 4 \cdot 1 \cdot 10}}{2 \cdot 1}
\]

Use the quadratic formula.

\[
x = \frac{12 \pm \sqrt{144 - 40}}{2} = \frac{12 \pm \sqrt{104}}{2} = \frac{12 \pm 2\sqrt{26}}{2}
\]

\[
= \frac{2 \cdot 6 \pm \sqrt{26}}{2} = 6 \pm \sqrt{26}
\]
continued

**Step 5**  **State the answer.** The slow chef’s time cannot be 0.9 since the fast chef’s time would then be 0.9 – 2 or –1.1. So the slow chef’s time working alone is 11.1 hr and the fast chef’s time working alone is 11.1 – 2 = 9.1 hr.

**Step 6**  **Check** that this value satisfies the original problem.
Objective 3

Solve an equation with radicals by writing it in quadratic form.
EXAMPLE 4

Solve $2x = \sqrt{x} + 1$.

$2x - 1 = \sqrt{x}$

$2x - 1 \quad 2 = \sqrt{x} \quad 2$

Isolate.

$4x^2 - 4x + 1 = x$

Square.

$4x^2 - 5x + 1 = 0$

$4x - 1 \quad x - 1 = 0$

$4x - 1 = 0 \quad$ or \quad $x - 1 = 0$

$x = \frac{1}{4} \quad$ or \quad $x = 1$
continued

Check both proposed solutions in the original equation,

\[
\begin{align*}
&x = \frac{1}{4} \quad \text{or} \quad x = 1 \\
&2x = \sqrt{x} + 1 \quad \quad 2x = \sqrt{x} + 1 \\
&\frac{1}{2} = \frac{1}{2} + 1 \\
&2 = 2
\end{align*}
\]

False \quad \quad \quad True

The solution set is \(1\).
Objective 4

Solve an equation that is quadratic in form by substitution.
EXAMPLE 5

Solve.

a. \(9x^4 - 37x^2 + 4 = 0\)

Let \(y = x^2\), so \(y^2 = (x^2)^2 = x^4\)

\[9y^2 - 37y + 4 = 0\]

\[y - 4 \quad 9y - 1 = 0\]

\[y - 4 = 0 \quad \text{or} \quad 9y - 1 = 0\]

\[y = 4 \quad \text{or} \quad y = \frac{1}{9}\]
continued

To find $x$, substitute $x^2$ for $y$.

$$x^2 = 4 \quad \text{or} \quad x^2 = \frac{1}{9}$$

$$x = \pm 2 \quad \text{or} \quad x = \pm \frac{1}{3}$$

Check

$$144 - 148 + 4 = 0 \quad \text{or} \quad \frac{1}{9} - \frac{37}{9} + 4 = 0$$

$$0 = 0 \quad \text{or} \quad 0 = 0$$

True \quad \text{True}

The solution set is $\left\{ -2, -\frac{1}{3}, \frac{1}{3}, 2 \right\}$. 
b. $x^4 - 4x^2 = -2$

$x^4 - 4x^2 + 2 = 0$

Let $y = x^2$, so $y^2 = (x^2)^2 = x^4$.

$$y^2 - 4y + 2 = 0$$

$a = 1$, $b = -4$, $c = 2$

$$y = \frac{-(-4) \pm \sqrt{(-4)^2 - 4 \cdot 1 \cdot 2}}{2 \cdot 1} = \frac{4 \pm \sqrt{16 - 8}}{2} = \frac{4 \pm \sqrt{8}}{2}$$

$$= \frac{4 \pm 2\sqrt{2}}{2} = \frac{2 \cdot 2 \pm \sqrt{2}}{2} = 2 \pm \sqrt{2}$$
continued

To find $x$, substitute $x^2$ for $y$.  

$$x^2 = 2 \pm \sqrt{2}$$

$$x = \pm \sqrt{2} \pm \sqrt{2}$$

Check

$$2 + \sqrt{2}^2 - 4 \ 2 + \sqrt{2} = -2$$

$$2 - \sqrt{2}^2 - 4 \ 2 - \sqrt{2} = -2$$

$$2 + \sqrt{2}^2 - 4 \ 2 + \sqrt{2} = -2$$

$$2 - \sqrt{2}^2 - 4 \ 2 - \sqrt{2} = -2$$

$$4 + 4\sqrt{2} + 2 - 8 - 4\sqrt{2} = -2$$

$$4 - 4\sqrt{2} + 2 - 8 + 4\sqrt{2} = -2$$

$$-2 = -2$$

$$-2 = -2$$

True  True

The solution set is \( \sqrt{2} + \sqrt{2}, -\sqrt{2} + \sqrt{2}, \sqrt{2} - \sqrt{2}, -\sqrt{2} - \sqrt{2} \).
EXAMPLE 6

Solve.

a. \( 5 \ (x + 3)^2 + 9 \ (x + 3) = 2 \)

Let \( y = x + 3 \), so the equation becomes:

\[
5y^2 - 9y = 2
\]

\[
5y - 1 \quad y + 2 = 0
\]

\[
5y - 1 = 0 \quad \text{or} \quad y + 2 = 0
\]

\[
y = \frac{1}{5} \quad \text{or} \quad y = -2
\]
continued

To find \( x \), substitute \( x + 3 \) for \( y \).

\[
x + 3 = \frac{1}{5} \quad \text{or} \quad x + 3 = -2
\]

\[
x = -\frac{14}{5} \quad \text{or} \quad x = -5
\]

Check

\[
\frac{1}{5} + \frac{9}{5} = 2 \quad \text{or} \quad 20 - 18 = 2
\]

\[
2 = 2 \quad \text{True} \quad \text{True}
\]

The solution set is \( \left\{ -5, -\frac{14}{5} \right\} \).
b. \(4x^{2/3} = 3x^{1/3} + 1\)

Let \(y = x^{1/3}\), so \(y^2 = (x^{1/3})^2 = x^{2/3}\).

\[4y^2 = 3y + 1\]
\[4y^2 - 3y - 1 = 0\]

\[4y + 1 = 0 \quad \text{or} \quad y - 1 = 0\]

\[4y + 1 = 0 \quad \text{or} \quad y - 1 = 0\]

\[y = -\frac{1}{4} \quad \text{or} \quad y = 1\]
continued

To find \( x \), substitute \( x^{1/3} \) for \( y \).

\[
x^{1/3} = \frac{1}{4} \quad \text{or} \quad x^{1/3} = 1
\]

\[
x^{1/3} = \left(\frac{1}{4}\right)^3 \quad \text{or} \quad x^{1/3} = 1^3
\]

\[
x = -\frac{1}{64} \quad \text{or} \quad x = 1
\]

Check

\[
\frac{1}{4} = -\frac{3}{4} + 1 \quad \quad \quad 4 = 3 + 1
\]

\[
\frac{1}{4} = \frac{1}{4} \quad \quad \quad 4 = 4 \quad \quad \quad \text{True}
\]

True \quad 2 = 2

The solution set is \( \left\{-\frac{1}{64}, 1\right\} \).
9.4 Formulas and Further Applications

1. Solve formulas for variables involving squares and square roots.

2. Solve applied problems using the Pythagorean formula.

3. Solve applied problems using area formulas.

4. Solve applied problems using quadratic functions as models.
Objective 1

Solve formulas for variables involving squares and square roots.
EXAMPLE 1

Solve each formula for the given variable. Keep in the answer in part a.

a. Solve $A = \pi r^2$ for $r$.

$$\frac{A}{\pi} = r^2$$

$$r = \pm \sqrt{\frac{A}{\pi}}$$
b. Solve \( s = 30 \sqrt{\frac{a}{p}} \) for \( a \).

\[
s^2 = 900 \cdot \frac{a}{p}
\]

Square both sides.

\[
ps^2 = 900a
\]

Multiply by \( p \).

\[
\frac{ps^2}{900} = a
\]

Divide by 900.
EXAMPLE 2

Solve \(2t^2 - 5t + k = 0\) for \(t\).

Use \(a = 2\), \(b = -5\), and \(c = k\) in the quadratic formula.

\[
t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]

\[
t = \frac{-(-5) \pm \sqrt{(-5)^2 - 4 \cdot 2 \cdot k}}{2 \cdot 2}
\]

\[
t = \frac{5 \pm \sqrt{25 - 8k}}{4}
\]

The solutions are \(t = \frac{5 + \sqrt{25 - 8k}}{4}\) and \(t = \frac{5 - \sqrt{25 - 8k}}{4}\).
Objective 2

Solve applied problems using the Pythagorean formula.
EXAMPLE 3

A ladder is leaning against a house. The distance from the bottom of the ladder to the house is 5 ft. The distance from the top of ladder to the ground is 1 ft less than the length of the ladder. How long is the ladder?

Step 1  Read the problem carefully.

Step 2  Assign the variable. Let $x =$ the length of the ladder. Then $x - 1 =$ the distance from the top of the ladder to the ground.
Step 3  Write an equation. The wall of the house is perpendicular to the ground, so this is a right triangle. Use the Pythagorean formula.

\[ a^2 + b^2 = c^2 \]

\[ 5^2 + (x-1)^2 = x^2 \]

Step 4  Solve.

\[ 25 + x^2 - 2x + 1 = x^2 \]

\[ 26 = 2x \]

\[ 13 = x \]
Step 5 State the answer. The length of the ladder is 13 feet and the distance of the top of the ladder to the ground is 12 feet.

Step 6 Check.

\[ 5^2 + 12^2 = 13^2 \] and 12 is one less than 13, as required.
Objective 3

Solve applied problems using area formulas.
EXAMPLE 4

Suppose the pool is 20 ft by 40 ft. The homeowner wants to plant a strip of grass around the edge of the pool. There is enough seed to cover 700 ft\(^2\). How wide should the grass strip be?

**Step 1** Read the problem carefully.

**Step 2** Assign the variable. Let \(x\) = the width of the grass strip
Step 3  Write an equation. The width of the larger rectangle is $20 + 20x$, and the length is $40 + 2x$.

Area of the rectangle $- \text{area of pool} = \text{area of grass}$

$$20 + 2x \quad 40 + 2x \quad - \quad 20 \quad 40 \quad = \quad 700$$

Step 4  Solve.  \[800 + 120x + 4x^2 - 800 = 700\]

\[4x^2 + 120x - 700 = 0\]

\[x^2 + 30x - 175 = 0\]

\[x + 35 \quad x - 5 = 0\]

\[x + 35 = 0 \quad \text{or} \quad x - 5 = 0\]

\[x = -35 \quad \text{or} \quad x = 5\]
Step 5  State the answer. The width cannot be –35, so the grass strip should be 5 feet wide.

Step 6  Check. If \( x = 5 \), then the area of the large rectangle is \((40 + 2 \cdot 5) = 50 \cdot 30 = 1500 \text{ ft}^2\). The area of the pool is \(40 \cdot 20 = 800 \text{ ft}^2\). So, the area of the grass strip is \(1500 - 800 = 700 \text{ ft}^2\), as required. The answer is correct.
Objective 4

Solve applied problems using quadratic functions as models.
EXAMPLE 5

A ball is projected upward from the ground. Its distance in feet from the ground at t seconds is

\[ s(t) = -16t^2 + 64t. \]

At what time will the ball be 32 feet from the ground.

\[ s(t) = -16t^2 + 64t \]

\[ 32 = -16t^2 + 64t \]

\[ 16t^2 - 64t + 32 = 0 \]

\[ t^2 - 4t + 2 = 0 \]
Use \( a = 1 \), \( b = -4 \), and \( c = 2 \) in the quadratic formula.

\[
t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]

\[
t = \frac{-(-4) \pm \sqrt{(-4)^2 - 4 \cdot 1 \cdot 2}}{2 \cdot 1} = \frac{4 \pm \sqrt{16 - 8}}{2} = \frac{4 \pm \sqrt{8}}{2} = 2 \pm \sqrt{2}
\]

The solutions are \( t = 2 + \sqrt{2} \approx 3.4 \) or \( 2 - \sqrt{2} \approx 0.6 \).

The ball will be at a height of 32 ft at about 0.6 seconds and 3.4 seconds.
EXAMPLE 6

Refer to the quadratic function in Example 6 on page 572 of your text;

a. Find the number of bankruptcy filings in 2000.

For 2000, \( x = 2000 - 1990 = 10 \).

\[ f(x) = 3.37x^2 - 28.6x + 133 \]

\[ f(10) = 3.37 \times 10^2 - 28.6 \times 10 + 133 \]

\[ = 184 \]

According to the model, there were 184 bankruptcy filings in 2000.
b. In what year(s) were there about 100 company bankruptcy filings?

\[
f(x) = 3.37x^2 - 28.6x + 133
\]

100 = 3.37x^2 - 28.6x + 133

Let \( f(x) = 100 \).

0 = 3.37x^2 - 28.6x + 33
Use $a = 3.37$, $b = -28.6$, and $c = 33$ in the quadratic formula.

$$x = \frac{-28.6 \pm \sqrt{(-28.6)^2 - 4 \times 3.37 \times 33}}{2 \times 3.37}$$

$$x = \frac{28.6 \pm \sqrt{373.12}}{6.74}$$

$x \approx 7.109 \approx 7$ or $x \approx 1.377 \approx 1$.

The values 1 and 7 correspond to the years 1991 and 1997.
9.5 Graphs of Quadratic Functions

1. Graph a quadratic function.
2. Graph parabolas with horizontal and vertical shifts.
3. Use the coefficient of $x^2$ to predict the shape and direction in which a parabola opens.
4. Find a quadratic function to model data.
Objective

1

Graph a quadratic function.
The graph shown below is a graph of the simplest quadratic function, defined by $y = x^2$. This graph is called a parabola.

The point $(0, 0)$, the lowest point on the curve, is the vertex.
The vertical line through the vertex is the **axis** of the parabola, here $x = 0$.

A parabola is **symmetric about its axis**.
Quadratic Function

A function that can be written in the form

\[ f(x) = ax^2 + bx + c \]

for real numbers \( a, b, \) and \( c, \) with \( a \neq 0, \) is a quadratic function.

The graph of any quadratic function is a parabola with a vertical axis.
Objective 2

Graph parabolas with horizontal and vertical shifts.
Parabolas do not need to have their vertices at the origin.

The graph of

\[ F(x) = x^2 + k \]

is shifted, or translated \( k \) units vertically compared to \( f(x) = x^2 \).
EXAMPLE 1

Graph \( f(x) = x^2 + 3 \). Give the vertex, domain, and range.

The graph has the same shape as \( f(x) = x^2 \), but shifted up 3 units.

Make a table of points.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( x^2 + 3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-2)</td>
<td>7</td>
</tr>
<tr>
<td>(-1)</td>
<td>4</td>
</tr>
<tr>
<td>(0)</td>
<td>3</td>
</tr>
<tr>
<td>(1)</td>
<td>4</td>
</tr>
<tr>
<td>(2)</td>
<td>7</td>
</tr>
</tbody>
</table>

vertex \((0, 3)\)  
domain: \((-\infty, \infty)\)  
range: \([3, \infty)\)
Vertical Shift

The graph of $F(x) = x^2 + k$ is a parabola with the same shape as the graph of $f(x) = x^2$. The parabola is shifted $k$ units up if $k > 0$, and $|k|$ units down if $k < 0$. The vertex is $(0, k)$. 
EXAMPLE 2

Graph \( f(x) = (x + 2)^2 \). Give the vertex, axis, domain, and range.

The graph has the same shape as \( f(x) = x^2 \), but shifted 2 units to the left.

Make a table of points.

<table>
<thead>
<tr>
<th>( x )</th>
<th>((x + 2)^2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>−5</td>
<td>9</td>
</tr>
<tr>
<td>−4</td>
<td>4</td>
</tr>
<tr>
<td>−2</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>1</td>
<td>9</td>
</tr>
</tbody>
</table>

vertex \((-2, 0)\)
axis \(x = -2\)
domain: \((-\infty, \infty)\)
range: \([0, \infty)\)
Horizontal Shift

The graph of $F(x) = (x - h)^2$ is a parabola with the same shape as the graph of $f(x) = x^2$. The parabola is shifted $h$ units horizontally: $h$ units to the right if $h > 0$, and $|h|$ units to the left if $h < 0$. The vertex is $(h, 0)$. 
EXAMPLE 3

Graph \( f(x) = (x - 2)^2 + 1 \). Give the vertex, axis, domain, and range.

The graph has the same shape as \( f(x) = x^2 \), but shifted 2 units to the right and 3 unit up. Make a table of points.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>5</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
</tr>
</tbody>
</table>

vertex (2, 1)
axis \( x = 2 \)
domain: \( (-\infty, \infty) \)
range: \( [1, \infty) \)
Vertex and Axis of Parabola

The graph of $F(x) = (x - h)^2 + k$ is a parabola with the same shape as the graph of $f(x) = x^2$, but with vertex $(h, k)$. The axis is the vertical line $x = h$. 
Objective 3

Use the coefficient of $x^2$ to predict the shape and direction in which a parabola opens.
EXAMPLE 4

Graph \( f(x) = -2x^2 - 3 \). Give the vertex, axis, domain, and range.

The coefficient \((-2)\) affects the shape of the graph; the 2 makes the parabola narrower.

The negative sign makes the parabola open down.

The graph is shifted down 3 units.
continued

Graph $f(x) = -2x^2 - 3$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-2$</td>
<td>$-11$</td>
</tr>
<tr>
<td>$-1$</td>
<td>$-5$</td>
</tr>
<tr>
<td>$0$</td>
<td>$-3$</td>
</tr>
<tr>
<td>$1$</td>
<td>$-$</td>
</tr>
<tr>
<td>$2$</td>
<td>$-11$</td>
</tr>
</tbody>
</table>

vertex $(0, -3)$

axis $x = 0$   domain: $(-\infty, \infty)$   range: $(-\infty, -3]$
General Principles

1. The graph of the quadratic function defined by
   \[ F(x) = a(x - h)^2 + k, \ a \neq 0, \]
   is a parabola with vertex \((h, k)\) and the vertical line \(x = h\) as axis.

2. The graph opens up if \(a\) is positive and down if \(a\) is negative.

3. The graph is wider than that of \(f(x) = a^2\) if \(0 < |a| < 1\). The graph is narrower than that of \(f(x) = x^2\) if \(|a| > 1\).
EXAMPLE

Decide whether each parabola opens up or down and whether it is wider or narrower than the graph of $f(x) = x^2$.

<table>
<thead>
<tr>
<th>$f(x)$</th>
<th>Opens</th>
<th>Wider/Narrower</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a. f(x) = -\frac{2}{3}x^2$</td>
<td>down</td>
<td>wider</td>
</tr>
<tr>
<td>$b. f(x) = \frac{3}{4}x^2 + 1$</td>
<td>up</td>
<td>wider</td>
</tr>
<tr>
<td>$c. f(x) = -2x^2 - 3$</td>
<td>down</td>
<td>narrower</td>
</tr>
<tr>
<td>$d. f(x) = 3x^2 + 2$</td>
<td>up</td>
<td>narrower</td>
</tr>
</tbody>
</table>
EXAMPLE 5

Graph \( f(x) = \frac{1}{2}(x - 2)^2 + 1 \).

Parabola opens up.
Narrower than \( f(x) = x^2 \)
Vertex: \((2, 1)\)
axis \( x = 2 \)
domain: \((−∞, ∞)\)
range: \([1, ∞)\)
Objective 4

Find a quadratic function to model data.
EXAMPLE 6

The number of higher order multiple births in the United States is rising. Let \( x \) represent the number of years since 1970 and \( y \) represent the rate of higher-order multiples born per 100,000 births since 1971. The data are shown in the following table.

Use the data points \((1, 29.1), (6, 35), (26, 152.6)\) to find a quadratic equation model for the data.

Use \( ax^2 + bx + c \)

\[(1, 29.1) = a(1)^2 + b(1) + c = 29.1\]
\[(6, 35) = a(6)^2 + b(6) + c = 35\]
\[(26, 152.6) = a(26)^2 + b(26) + c = 152.6\]
continued

Simplify the system:

\[ a + b + c = 29.1 \] \hspace{1cm} (1)
\[ 36a + 6b + c = 35 \] \hspace{1cm} (2)
\[ 676a + 26b + c = 152.6 \] \hspace{1cm} (3)

To eliminate \( c \), multiply equation (1) by \(-1\) and add the result to equation (2).

\[ -a - b - c = -29.1 \] \hspace{1cm} \(-1 \times (1)\)
\[ 36a + 6b + c = 35 \] \hspace{1cm} (2)

\[ 35a + 5b = 5.9 \] \hspace{1cm} (4)
continued

To eliminate \(c\) again, multiply equation (2) by \(-1\) and add the result to equation (3).

\[
-36a - 6b - c = -35 \quad -1 \times (2)
\]
\[
676a + 26b + c = 152.6 \quad (3)
\]
\[
\frac{640a + 20b}{640a + 20b} = 117.6 \quad (5)
\]

To eliminate \(b\), multiply equation (4) by \(-4\) and add the result to equation (5).

\[
-140a - 20b = -23.6 \quad -4 \times (4)
\]
\[
\frac{640a + 20b}{640a + 20b} = 117.6 \quad (5)
\]
\[
500a = 94
\]
\[
a = 94/500 = 0.188
\]
To find $b$, substitute 0.188 for $a$ in equation (4).

\[ 35a + 5b = 5.9 \]
\[ 35(0.188) + 5b = 5.9 \]
\[ 6.58 + 5b = 5.9 \]
\[ 5b = -0.68 \]
\[ b = -0.136 \]

To find $c$, use equation (1).

\[ a + b + c = 29.1 \quad (1) \]
\[ c = 29.1 - a - b \]
\[ c = 29.1 - 0.188 - (-0.136) = 29.048 \]

The quadratic model using the three points is:

\[ y = 0.188x^2 - 0.136x + 29.048 \]
More About Parabolas and Their Applications

1. Find the vertex of a vertical parabola.
2. Graph a quadratic function.
3. Use the discriminant to find the number of $x$-intercepts of a parabola with a vertical axis.
4. Use quadratic functions to solve problems involving maximum or minimum value.
5. Graph parabolas with horizontal axes.
Objective 1

Find the vertex of a vertical parabola.
When the equation of a parabola is given in the form $f(x) = ax^2 + bx + c$, we need to locate the vertex to sketch an accurate graph. There are two ways to do this:

1. Complete the square.
2. Use a formula derived by completing the square.
EXAMPLE 1

Find the vertex of the graph of \( f(x) = x^2 + 4x - 9 \).

We need to complete the square.

\[
\begin{align*}
f(x) &= x^2 + 4x - 9 \\
&= (x^2 + 4x + 4 - 4) - 9 \\
&= (x^2 + 4x + 4) - 4 - 9 \\
&= (x + 2)^2 - 13
\end{align*}
\]

The vertex of the parabola is \((-2, -13)\).
EXAMPLE 2

Find the vertex of the graph of \( f(x) = 2x^2 - 4x + 1 \).

We need to complete the square, factor out 2 from the first two terms.

\[
f(x) = 2x^2 - 4x + 1
\]

\[
f(x) = 2(x^2 - 2x) + 1
\]

\[
= 2(x^2 - 2x + 1 - 1) + 1
\]

\[
= 2(x^2 - 2x + 1) + 2(-1) + 1
\]

\[
= 2(x^2 - 2x + 1) - 2 + 1
\]

\[
= 2(x - 1)^2 - 1
\]

The vertex of the parabola is \((1, -1)\).
### Vertex Formula

The graph of the quadratic function defined by \( f(x) = ax^2 + bx + c \ (a \neq 0) \) has vertex

\[
\left( \frac{-b}{2a}, f\left( \frac{-b}{2a} \right) \right),
\]

and the axis of the parabola is the line

\[
x = \frac{-b}{2a}.
\]
EXAMPLE 3

Use the vertex formula to find the vertex of the graph of \( f(x) = -2x^2 + 3x - 1 \).

\( a = -2, \ b = 3, \) and \( c = -1 \).

The \( x \)-coordinate:

\[
\frac{-b}{2a} = \frac{-3}{2(-2)} = \frac{-3}{-4} = \frac{3}{4}
\]

The \( y \)-coordinate:

\[
f \left( \frac{3}{4} \right) = -2 \left( \frac{3}{4} \right)^2 + 3 \left( \frac{3}{4} \right) - 1
\]

\[
= -2 \left( \frac{9}{16} \right) + \left( \frac{9}{4} \right) - 1
\]

\[
= -\frac{9}{8} + \frac{18}{8} - \frac{8}{8} = \frac{1}{8}
\]

The vertex of the parabola is \( \left( \frac{3}{4}, \frac{1}{8} \right) \).
Objective 2

Graph parabolas with horizontal and vertical shifts.
Graphing a Quadratic Function $y = f(x)$

**Step 1** Determine whether the graph opens up or down. If $a > 0$, the parabola opens up; if $a < 0$, it opens down.

**Step 2** Find the vertex. Use either the vertex formula or completing the square.

**Step 3** Find any intercepts. To find the $x$-intercepts (if any), solve $f(x) = 0$. To find the $y$-intercept, evaluate $f(0)$.

**Step 4** Complete the graph. Plot the points found so far. Find and plot additional points as needed, using symmetry about the axis.
EXAMPLE 4

Graph \( f(x) = x^2 - 6x + 5 \). Give the vertex, axis, domain, and range.

**Step 1** The graph opens up since \( a = 1 \), which is \( > 0 \).

**Step 2** Find the vertex. Complete the square.

\[
\begin{align*}
f(x) &= x^2 - 6x + 5 \\
&= x^2 - 6x + 9 - 9 + 5 \\
&= (x - 3)^2 - 4
\end{align*}
\]

The vertex is at \((3, -4)\).
continued

Graph $f(x) = x^2 - 6x + 5$.

**Step 3** Find any $x$-intercepts. Let $f(x) = 0$

$$0 = x^2 - 6x + 5$$

$$0 = (x - 5)(x - 1)$$

$x - 5 = 0$ or $x - 1 = 0$

$x = 5$ or $x = 1$

The $x$-intercepts are $(5, 0)$ and $(1, 0)$.

Find the $y$-intercept. Let $x = 0$.

$$f(x) = 0^2 - 6(0) + 5 = 5$$

The $y$-intercept is $(0, 5)$. 
continued

Graph $f(x) = x^2 - 6x + 5$.

**Step 4** Plot the points:

- Vertex: $(3, -4)$.
- $x$-intercepts: $(5, 0)$ and $(1, 0)$
- $y$-intercept: $(0, 5)$
- Axis of symmetry: $x = 3$

By symmetry $(6, 5)$ is also another point on the graph.

- Domain: $(-\infty, \infty)$
- Range: $[-4, \infty)$
Objective 3

Use the discriminant to find the number of $x$-intercepts of a parabola with a vertical axis.
You can use the discriminant to determine the number of $x$-intercepts.

$$b^2 - 4ac < 0$$
No $x$-intercepts

$$b^2 - 4ac = 0$$
One $x$-intercept

$$b^2 - 4ac > 0$$
Two $x$-intercepts
EXAMPLE 5

Find the discriminant and use it to determine the number of $x$-intercepts of the graph of each quadratic function.

a. $f(x) = -3x^2 - x + 2$

$a = -3, \ b = -1, \ c = 2$

$b^2 - 4ac = (-1)^2 - 4(-3)(2)$

$= 1 + 24$

$= 25$

The discriminant is positive, the graph has two $x$-intercepts.
continued

b. $f(x) = x^2 - x + 1$

The discriminant is negative, the graph has no $x$-intercepts.

$c. f(x) = x^2 - 8x + 16$

$$a = 1, \quad b = -8, \quad c = 16$$

$$b^2 - 4ac = (-8)^2 - 4(1)(16) = 64 - 64 = 0$$

The discriminant is 0, the graph has one $x$-intercept.
Objective 4

Use quadratic functions to solve problems involving maximum or minimum value.
EXAMPLE 6

A farmer has 100 ft of fencing. He wants to put a fence around a rectangular field next to a building. Find the maximum area he can enclose, and the dimensions of the field when the area is maximized.

Let \( x \) = the width of the field

\[ x + x + \text{length} = 100 \]
\[ 2x + \text{length} = 100 \]

\[ \text{length} = 100 - 2x \]
continued

Area = width \times length

\[ A(x) = x(100 - 2x) \]
\[ = 100x - 2x^2 \]

Determine the vertex:
\[ x = \frac{-b}{2a} = \frac{-100}{2(-2)} = 25 \]
\[ a = -2, \ b = 100, \ c = 0 \]

y-coordinate: \[ f(25) = -2(25)^2 + 100(25) \]
\[ = -2(625) + 2500 \]
\[ = -1250 + 2500 \]
\[ = 1250 \]

Vertex: (25, 1250)
Parabola opens down with vertex (25, 1250).

The vertex shows that the maximum area will be 1250 square feet.

The area will occur if the width, $x$, is 25 feet and the length is $100 - 2x$, or 50 feet.
EXAMPLE 7

A toy rocket is launched from the ground so that its distance above the ground after \( t \) seconds is

\[ s(t) = -16t^2 + 208t \]

Find the maximum height it reaches and the number of seconds it takes to reach that height.

Find the vertex of the function.

\[ a = -16, \quad b = 208 \]

\[ x = \frac{-b}{2a} = \frac{-208}{2(-16)} = \frac{13}{2} = 6.5 \]
continued

Find the y-coordinate.

\[ f \left( \frac{13}{2} \right) = -16 \left( \frac{13}{2} \right)^2 + 208 \left( \frac{13}{2} \right) \]
\[ = -16 \left( \frac{169}{4} \right) + 1352 \]
\[ = -676 + 1352 \]
\[ = 676 \]

The toy rocket reaches a maximum height of 676 feet in 6.5 seconds.
Objective 5

Graph parabolas with horizontal axes.
Graph of a Parabola with Horizontal Axis

The graph of

\[ x = ay^2 + by + c \text{ or } x = a(y - k)^2 + h \]

is a parabola with vertex \((h, k)\) and horizontal line \(y = k\) as axis. The graph opens to the right if \(a > 0\) and to the left if \(a < 0\).
EXAMPLE 8

Graph \( x = (y + 1)^2 - 4 \). Give the vertex, axis, domain, and range.

Vertex: \((-4, -1)\)

Opens: right since \( a > 1 \)

Axis: \( y = -1 \)

\[
\begin{array}{|c|c|}
\hline
x & f(x) \\
\hline
-3 & 0 \\
-3 & -2 \\
0 & 1 \\
0 & -3 \\
\hline
\end{array}
\]

Domain: \([-4, \infty)\) \quad Range: \((-\infty, \infty)\)
EXAMPLE 9

Graph $x = -y^2 + 2y + 5$. Give the vertex, axis, domain, and range.

Complete the square

$$x = -(y^2 - 2y) + 5$$
$$= -(y^2 - 2y + 1) - (-1) + 5$$
$$= -(y - 1)^2 + 6$$

Vertex: $(6, 1)$

Opens: left, since $a < 1$

Axis: $y = 1$

Domain: $[-\infty, 6]$

Range: $(-\infty, \infty)$
### Graphs of Parabolas

<table>
<thead>
<tr>
<th>Equation</th>
<th>Graph</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y = ax^2 + bx + c )</td>
<td><img src="image1.png" alt="Graph 1" /></td>
</tr>
<tr>
<td>( y = a(x - h)^2 + k )</td>
<td><img src="image2.png" alt="Graph 2" /></td>
</tr>
<tr>
<td>( x = ay^2 + by + c )</td>
<td><img src="image3.png" alt="Graph 3" /></td>
</tr>
<tr>
<td>( x = a(y - k)^2 + h )</td>
<td><img src="image4.png" alt="Graph 4" /></td>
</tr>
</tbody>
</table>

- When \( a > 0 \), these graphs represent functions.
- When \( a < 0 \), these graphs are not graphs of functions.
Quadratic and Rational Inequalities

1. Solve quadratic inequalities.
2. Solve polynomial inequalities of degree 3 or greater.
3. Solve rational inequalities.
A **quadratic inequality** can be written in the form

\[ ax^2 + bx + c < 0 \text{ or } ax^2 + bx + c > 0, \]

where \( a, b, \) and \( c \) are real numbers, with \( a \neq 0 \).
Objective 1

Solve quadratic inequalities.
EXAMPLE 1

Use the graph to solve each quadratic inequality.

a. \( x^2 + 6x + 8 > 0 \)

Find the \( x \)-intercepts.

\( x^2 + 6x + 8 = 0 \)

\( (x + 2)(x + 4) = 0 \)

\( x + 2 = 0 \) or \( x + 4 = 0 \)

\( x = -2 \) or \( x = -4 \)

Notice from the graph that \( x \)-values less than \( -4 \) or greater than \( -2 \) result in \( y \)-values greater than \( 0 \).

The solution set is \( -\infty, -4 \cup -2, \infty \).
b. \( x^2 + 6x + 8 < 0 \)

Find the \( x \)-intercepts.

\[
x^2 + 6x + 8 = 0
\]

\[
(x + 2)(x + 4) = 0
\]

\[
x + 2 = 0 \quad \text{or} \quad x + 4 = 0
\]

\[
x = -2 \quad \text{or} \quad x = -4
\]

Notice from the graph that \( x \)-values between \(-4\) and \(-2\) result in \( y \)-values less than 0.

The solution set is \(-4, -2\).
EXAMPLE 2

Solve and graph the solution set. \( 2x^2 + 3x \geq 2 \)

Use factoring to solve the quadratic equation.

\[
2x^2 + 3x - 2 = 0
\]

\[
(2x - 1)(x + 2) = 0
\]

\[
2x - 1 = 0 \quad \text{or} \quad x + 2 = 0
\]

\[
x = \frac{1}{2} \quad \text{or} \quad x = -2
\]

The numbers divide a number line into three intervals.
Choose a number from each interval to substitute in the inequality.

Interval A: Let $x = -3$.

\[
2(-3)^2 + 3(-3) \geq 2
\]
\[
18 - 9 \geq 2
\]
\[
9 \geq 2 \quad \text{True}
\]

Interval B: Let $x = 0$.

\[
2(0)^2 + 3(0) \geq 2
\]
\[
0 + 0 \geq 2
\]
\[
0 \geq 2 \quad \text{False}
\]
continued

Interval C: Let $x = 1$.

$$2(1)^2 + 3(1) \geq 2$$

$$2 + 3 \geq 2$$

$$5 \geq 2 \quad \text{True}$$

The numbers in Intervals A and C are solutions. The numbers $-2$ and $\frac{1}{2}$ are included because of the $\geq$.

Solution set: $-\infty, -2 \cup \frac{1}{2}, \infty$
Solving a Quadratic Inequality

*Step 1* Write the inequality as an equation and solve it.

*Step 2* Use the solutions from Step 1 to determine intervals. Graph the numbers found in Step 1 on a number line. These numbers divide the number line into intervals.

*Step 3* Find the intervals that satisfy the inequality. Substitute a test number from each interval into the original inequality to determine the intervals that satisfy the inequality. All numbers in those intervals are in the solution set. A graph of the solution set will usually look like one of these. (Square brackets might be used instead of parentheses.)

![Graph of solution set]

*Step 4* Consider the endpoints separately. The numbers from Step 1 are included in the solution set if the inequality symbol is \( \leq \) or \( \geq \); they are not included if it is \( < \) or \( > \).
EXAMPLE 3

Solve.

a. \((3x - 2)^2 > -2\)

The square of any real number is always greater than or equal to 0, so any real number satisfies this inequality. The solution set is the set of all real numbers, \((-\infty, \infty)\).

b. \((3x - 2)^2 < -2\)

The square of a real number is never negative, there is no solution for this inequality. The solution set is \(\emptyset\).
Objective 2

Solve polynomial inequalities of degree 3 or greater.
EXAMPLE 4

Solve and graph the solution set.

\((2x + 1)(3x - 1)(x + 4) > 0\)

Set each factored polynomial equal to 0 and solve the equation.

\((2x + 1)(3x - 1)(x + 4) = 0\)

\[2x + 1 = 0 \quad \text{or} \quad 3x - 1 = 0 \quad \text{or} \quad x + 4 = 0\]

\[x = -\frac{1}{2} \quad x = \frac{1}{3} \quad x = -4\]
continued

Substitute a test number from each interval in the original inequality.

<table>
<thead>
<tr>
<th>Interval</th>
<th>Test Number</th>
<th>Test of Inequality</th>
<th>True or False?</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>–5</td>
<td>–144 &gt; 0</td>
<td>False</td>
</tr>
<tr>
<td>B</td>
<td>–2</td>
<td>42 &gt; 0</td>
<td>True</td>
</tr>
<tr>
<td>C</td>
<td>0</td>
<td>–4 &gt; 0</td>
<td>False</td>
</tr>
<tr>
<td>D</td>
<td>1</td>
<td>30 &gt; 0</td>
<td>True</td>
</tr>
</tbody>
</table>

The numbers in Intervals B and D, not including the endpoints are solutions.

Solution set: \(-4, \frac{-1}{2} \cup \frac{1}{3}, \infty\)
Objective 3

Solve rational inequalities.
Solving a Rational Inequality

**Step 1** Write the inequality so that 0 is on one side and there is a single fraction on the other side.

**Step 2** Determine the numbers that make the numerator or denominator equal to 0.

**Step 3** Divide a number line into intervals. Use the numbers from Step 2.

**Step 4** Find the intervals that satisfy the inequality. Test a number from each interval by substituting it into the original inequality.

**Step 5** Consider the endpoints separately. Exclude any values that make the denominator 0.
EXAMPLE 5

Solve and graph the solution set. \( \frac{2}{x-4} < 3 \)

Write the inequality so that 0 is on one side.

\[
\frac{2}{x-4} - 3 < 0
\]

\[
\frac{2 - 3(x-4)}{x-4} < 0
\]

The number \( \frac{14}{3} \) makes the numerator 0, and 4 makes the denominator 0. These two numbers determine three intervals.
Test a number from each interval.

<table>
<thead>
<tr>
<th>Interval</th>
<th>Test Number</th>
<th>Test of Inequality</th>
<th>True or False?</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>0</td>
<td>-1/2 &lt; 3</td>
<td>True</td>
</tr>
<tr>
<td>B</td>
<td>13/3</td>
<td>6 &lt; 3</td>
<td>False</td>
</tr>
<tr>
<td>C</td>
<td>5</td>
<td>2 &lt; 3</td>
<td>True</td>
</tr>
</tbody>
</table>

The solution set includes numbers in Intervals A and C, excluding endpoints.

Solution set:  \(-\infty, 4 \cup \frac{14}{3}, \infty\)
EXAMPLE 6

Solve and graph the solution set. \( \frac{x + 2}{x - 1} \leq 5 \)

Write the inequality so that 0 is on one side.

\[
\frac{x + 2}{x - 1} - 5 \leq 0
\]

\[
\frac{x + 2 - 5(x - 1)}{x - 1} \leq 0
\]

\[
\frac{x + 2 - 5x + 5}{x - 1} \leq 0
\]

\[
\frac{-4x + 7}{x - 1} \leq 0
\]

The number 7/4 makes the numerator 0, and 1 makes the denominator 0. These two numbers determine three intervals.
Test a number from each interval.

<table>
<thead>
<tr>
<th>Interval</th>
<th>Test Number</th>
<th>Test of Inequality</th>
<th>True or False?</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>0</td>
<td>$-2 \leq 5$</td>
<td>True</td>
</tr>
<tr>
<td>B</td>
<td>$3/2$</td>
<td>$7 \leq 5$</td>
<td>False</td>
</tr>
<tr>
<td>C</td>
<td>2</td>
<td>$4 \leq 5$</td>
<td>True</td>
</tr>
</tbody>
</table>

The numbers in Intervals A and C are solutions. 1 is NOT in the solution set (since it makes the denominator 0), but $7/4$ is.

Solution set: $-\infty, 1 \cup \left[ \frac{7}{4}, \infty \right)$