In this chapter we will study kinematics, i.e., how objects move along a straight line.

The following parameters will be defined:

- Displacement
- Average velocity
- Average speed
- Instantaneous velocity
- Average and instantaneous acceleration

For constant acceleration we will develop the equations that give us the velocity and position at any time. In particular we will study the motion under the influence of gravity close to the surface of the Earth.

Finally, we will study a graphical integration method that can be used to analyze the motion when the acceleration is not constant.
Kinematics is the part of mechanics that describes the motion of physical objects. We say that an object moves when its position as determined by an observer changes with time.

In this chapter we will study a restricted class of kinematics problems. Motion will be along a straight line.

We will assume that the moving objects are “particles,” i.e., we restrict our discussion to the motion of objects for which all the points move in the same way.

The causes of the motion will not be investigated. This will be done later in the course.

Consider an object moving along a straight line taken to be the $x$-axis. The object’s position at any time $t$ is described by its coordinate $x(t)$ defined with respect to the origin $O$. The coordinate $x$ can be positive or negative depending whether the object is located on the positive or the negative part of the $x$-axis.
**Displacement.** If an object moves from position \( x_1 \) to position \( x_2 \), the change in position is described by the displacement

\[
\Delta x = x_2 - x_1
\]

For example if \( x_1 = 5 \text{ m} \) and \( x_2 = 12 \text{ m} \) then \( \Delta x = 12 - 5 = 7 \text{ m} \). The positive sign of \( \Delta x \) indicates that the motion is along the positive \( x \)-direction.

If instead the object moves from \( x_1 = 5 \text{ m} \) and \( x_2 = 1 \text{ m} \) then \( \Delta x = 1 - 5 = -4 \text{ m} \). The negative sign of \( \Delta x \) indicates that the motion is along the negative \( x \)-direction.

Displacement is a vector quantity that has both magnitude and direction. In this restricted one-dimensional motion the direction is described by the algebraic sign of \( \Delta x \).

**Note:** The actual distance for a trip is irrelevant as far as the displacement is concerned.

Consider as an example the motion of an object from an initial position \( x_1 = 5 \text{ m} \) to \( x = 200 \text{ m} \) and then back to \( x_2 = 5 \text{ m} \). Even though the total distance covered is 390 m the displacement then is \( \Delta x = 0 \). (2-3)
Average Velocity

One method of describing the motion of an object is to plot its position $x(t)$ as a function of time $t$. In the left picture we plot $x$ versus $t$ for an object that is stationary with respect to the chosen origin $O$. Notice that $x$ is constant. In the picture to the right we plot $x$ versus $t$ for a moving armadillo. We can get an idea of “how fast” the armadillo moves from one position $x_1$ at time $t_1$ to a new position $x_2$ at time $t_2$ by determining the average velocity between $t_1$ and $t_2$.

$$v_{\text{avg}} = \frac{x_2 - x_1}{t_2 - t_1} = \frac{\Delta x}{\Delta t}$$

Here $x_2$ and $x_1$ are the positions $x(t_2)$ and $x(t_1)$, respectively.

The time interval $\Delta t$ is defined as $\Delta t = t_2 - t_1$. The units of $v_{\text{avg}}$ are m/s.

Note: For the calculation of $v_{\text{avg}}$ both $t_1$ and $t_2$ must be given.
Graphical Determination of $v_{avg}$

On an $x$ versus $t$ plot we can determine $v_{avg}$ from the slope of the straight line that connects point $(t_1, x_1)$ with point $(t_2, x_2)$. In the plot below, $t_1=1$ s and $t_2=4$ s. The corresponding positions are: $x_1 = -4$ m and $x_2 = 2$ m.

$$v_{avg} = \frac{x_2 - x_1}{t_2 - t_1} = \frac{2 - (-4)}{4 - 1} = \frac{6 \text{ m}}{3 \text{ s}} = 2 \text{ m/s}$$

Average Speed $s_{avg}$

The average speed is defined in terms of the total distance traveled in a time interval $\Delta t$ (and not the displacement $\Delta x$ as in the case of $v_{avg}$).

$$s_{avg} = \frac{\text{total distance}}{\Delta t}$$

**Note:** The average velocity and the average speed for the same time interval $\Delta t$ can be quite different.
Instantaneous Velocity

The average velocity $v_{\text{avg}}$ determined between times $t_1$ and $t_2$ provides a useful description of “how fast” an object is moving between these two times. It is in reality a “summary” of its motion. In order to describe how fast an object moves at any time $t$ we introduce the notion of instantaneous velocity $v$ (or simply velocity). Instantaneous velocity is defined as the limit of the average velocity determined for a time interval $\Delta t$ as we let $\Delta t \to 0$.

$$v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}$$

From its definition instantaneous velocity is the first derivative of the position coordinate $x$ with respect to time. It is thus equal to the slope of the $x$ versus $t$ plot.

**Speed**

We define speed as the magnitude of an object’s velocity vector.

(2-6)
Average Acceleration

We define the average acceleration $a_{\text{avg}}$ between $t_1$ and $t_2$ as:

$$a_{\text{avg}} = \frac{v_2 - v_1}{t_2 - t_1} = \frac{\Delta v}{\Delta t}$$

Units: m/s$^2$

Instantaneous Acceleration

If we take the limit of $a_{\text{avg}}$ as $\Delta t \to 0$ we get the instantaneous acceleration $a$, which describes how fast the velocity is changing at any time $t$.

$$a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt}, \quad a = \frac{dv}{dt} = \frac{d}{dt} \left( \frac{dx}{dt} \right) = \frac{d^2 x}{dt^2}$$

The acceleration is the slope of the $v$ versus $t$ plot.

Note: The human body does not react to velocity but it does react to acceleration.
Motion with Constant Acceleration

Motion with $a = 0$ is a special case but it is rather common, so we will develop the equations that describe it.

$$a = \frac{dv}{dt} \rightarrow dv = adt.$$ If we integrate both sides of the equation we get:

$$\int dv = \int adt = a \int dt \rightarrow v = at + C.$$ Here $C$ is the integration constant.

$C$ can be determined if we know the velocity $v_0 = v(0)$ at $t = 0$:

$$v(0) = v_0 = (a)(0) + C \rightarrow C = v_0 \rightarrow \quad v = v_0 + at \quad \text{(eq. 1)}$$

$$v = \frac{dx}{dt} \rightarrow dx = vdt = (v_0 + at)dt = v_0 dt + atdt.$$ If we integrate both sides we get:

$$\int dx = \int v_0 dt + a \int t dt \rightarrow x = v_0 t + \frac{at^2}{2} + C'.$$ Here $C'$ is the integration constant.

$C'$ can be determined if we know the position $x_o = x(0)$ at $t = 0$:

$$x(0) = x_o = (v_0)(0) + \frac{a}{2} (0) + C' \rightarrow C' = x_o$$

$$x(t) = x_o + v_0 t + \frac{at^2}{2} \quad \text{(eq. 2)}$$

(2-8)
\[ v = v_0 + at \] (eq. 1); \[ x = x_0 + v_0 t + \frac{at^2}{2} \] (eq. 2)

If we eliminate the time \( t \) between equation 1 and equation 2 we get:
\[ v^2 - v_0^2 = 2a \ (x - x_0) \] (eq. 3)

Below we plot the position \( x(t) \), the velocity \( v(t) \), and the acceleration \( a \) versus time \( t \):

The \( x(t) \) versus \( t \) plot is a parabola that intercepts the vertical axis at \( x = x_0 \).

The \( v(t) \) versus \( t \) plot is a straight line with slope = \( a \) and intercept = \( v_0 \).

The acceleration \( a \) is a constant.
Free Fall

Close to the surface of the Earth all objects move toward the center of the Earth with an acceleration whose magnitude is constant and equal to 9.8 m/s\(^2\). We use the symbol \(g\) to indicate the acceleration of an object in free fall.

If we take the \(y\)-axis to point upward then the acceleration of an object in free fall \(a = -g\) and the equations for free fall take the form:

\[
\begin{align*}
  v &= v_0 - gt \quad \text{(eq. 1)} \\
  x &= x_0 + v_0 t - \frac{gt^2}{2} \quad \text{(eq. 2)} \\
  v^2 - v_0^2 &= -2g(x - x_0) \quad \text{(eq. 3)}
\end{align*}
\]

**Note:** Even though with this choice of axes \(a < 0\), the velocity can be positive (upward motion from point \(A\) to point \(B\)). It is momentarily zero at point \(B\). The velocity becomes negative on the downward motion from point \(B\) to point \(A\).

**Hint:** In a kinematics problem, always indicate the axis as well as the acceleration vector. This simple precaution helps to avoid algebraic sign errors.
Graphical Integration in Motion Analysis (nonconstant acceleration)

When the acceleration of a moving object is not constant we must use integration to determine the velocity $v(t)$ and the position $x(t)$ of the object. The integration can be done either using the analytic or the graphical approach:

$$a = \frac{dv}{dt} \rightarrow dv = adt \rightarrow \int_{t_0}^{t_1} dv = \int_{t_0}^{t_1} adt \rightarrow v_1 - v_0 = \int_{t_0}^{t_1} adt \rightarrow v_1 = v_0 + \int_{t_0}^{t_1} adt$$

$$\int_{t_0}^{t_1} adt = \text{Area under the } a \text{ versus } t \text{ curve between } t_0 \text{ and } t_1$$

$$(a)$$

$$v = \frac{dx}{dt} \rightarrow dx = vdt \rightarrow \int_{t_0}^{t_1} dx = \int_{t_0}^{t_1} vdt \rightarrow$$

$$x_1 - x_0 = \int_{t_0}^{t_1} vdt \rightarrow x_1 = x_0 + \int_{t_0}^{t_1} vdt$$

$$(b)$$

$$\int_{t_0}^{t_1} vdt = \text{Area under the } v \text{ versus } t \text{ curve between } t_0 \text{ and } t_1$$

(2-11)