Chapter 7

Kinetic Energy and Work

In this chapter we will introduce the following concepts:

Kinetic energy of a moving object
Work done by a force
Power

In addition, we will develop the work-kinetic energy theorem and apply it to solve a variety of problems.

This approach is an alternative approach to mechanics. It uses scalars such as work and kinetic energy rather than vectors such as velocity and acceleration. Therefore it is simpler to apply.
Kinetic Energy:
We define a new physical parameter to describe the state of motion of an object of mass $m$ and speed $v$.

We define its kinetic energy $K$ as

$$K = \frac{mv^2}{2}.$$

We can use the equation above to define the SI unit for work (the joule, symbol: $J$). An object of mass $m = 1$kg that moves with speed $v = 1$ m/s has a kinetic energy $K = 1$J.

Work: (symbol $W$)
If a force $F$ is applied to an object of mass $m$ it can accelerate it and increase its speed $v$ and kinetic energy $K$. Similarly $F$ can decelerate $m$ and decrease its kinetic energy.

We account for these changes in $K$ by saying that $F$ has transferred energy $W$ to or from the object. If energy is transferred to $m$ (its $K$ increases) we say that work was done by $F$ on the object ($W > 0$). If on the other hand, energy is transferred from the object (its $K$ decreases) we say that work was done by $m$ ($W < 0$).
Finding an Expression for Work:
Consider a bead of mass $m$ that can move without friction along a straight wire along the $x$-axis. A constant force $\vec{F}$ applied at an angle $\phi$ to the wire is acting on the bead.

We apply Newton's second law: $F_x = ma_x$. We assume that the bead had an initial velocity $\vec{v}_0$ and after it has traveled a distance $\vec{d}$ its velocity is $\vec{v}$. We apply the third equation of kinematics: $v^2 - v_0^2 = 2a_xd$. We multiply both sides by $m/2 \rightarrow \frac{m}{2} v^2 - \frac{m}{2} v_0^2 = \frac{m}{2} 2a_xd = \frac{m}{2} 2 F_x \frac{d}{m} = F_xd = F \cos \phi \cdot d$. $K_i = \frac{m}{2} v_0^2$.

$K_f = \frac{m}{2} v^2 \rightarrow$ The change in kinetic energy $K_f - K_i = Fd \cos \phi$.

Thus the work $W$ done by the force on the bead is given by $W = F_xd = Fd \cos \phi$.

$$W = Fd \cos \phi$$

W = \vec{F} \cdot \vec{d}$$

(7-3)
The unit of $W$ is the same as that of $K$, i.e., joules.

Note 1: The expressions for work we have developed apply when $F$ is constant.

Note 2: We have made the implicit assumption that the moving object is point-like.

Note 3: $W > 0$ if $0 < \phi < 90^\circ$, $W < 0$ if $90^\circ < \phi < 180^\circ$.

Net Work: If we have several forces acting on a body (say three as in the picture) there are two methods that can be used to calculate the net work $W_{\text{net}}$.

Method 1: First calculate the work done by each force: $W_A$ by force $\vec{F}_A$, $W_B$ by force $\vec{F}_B$, and $W_C$ by force $\vec{F}_C$. Then determine $W_{\text{net}} = W_A + W_B + W_C$.

Method 2: Calculate first $\vec{F}_{\text{net}} = \vec{F}_A + \vec{F}_B + \vec{F}_C$; then determine $W_{\text{net}} = \vec{F} \cdot \vec{d}$. (7-4)
Work-Kinetic Energy Theorem

We have seen earlier that \( K_f - K_i = W_{\text{net}} \).

We define the change in kinetic energy as \( \Delta K = K_f - K_i \). The equation above becomes the work-kinetic energy theorem:

\[
\Delta K = K_f - K_i = W_{\text{net}}
\]

\[
\begin{bmatrix}
\text{Change in the kinetic energy of a particle}
\end{bmatrix} = \begin{bmatrix}
\text{net work done on the particle}
\end{bmatrix}
\]

The work-kinetic energy theorem holds for both positive and negative values of \( W_{\text{net}} \).

If \( W_{\text{net}} > 0 \) → \( K_f - K_i > 0 \) → \( K_f > K_i \)

If \( W_{\text{net}} < 0 \) → \( K_f - K_i < 0 \) → \( K_f < K_i \)

(7-5)
Consider a tomato of mass $m$ that is thrown upward at point $A$ with initial speed $v_0$. As the tomato rises, it slows down by the gravitational force $F_g$ so that at point $B$ it has a smaller speed $v$. The work $W_g (A \rightarrow B)$ done by the gravitational force on the tomato as it travels from point $A$ to point $B$ is

$$W_g (A \rightarrow B) = mgd \cos 180° = -mgd.$$ 

The work $W_g (B \rightarrow A)$ done by the gravitational force on the tomato as it travels from point $B$ to point $A$ is

$$W_g (B \rightarrow A) = mgd \cos 0° = mgd.$$ 

(7-6)
Work Done by a Force in Lifting an Object:
Consider an object of mass $m$ that is lifted by a force $F$ from point $A$ to point $B$. The object starts from rest at $A$ and arrives at $B$ with zero speed. The force $F$ is not necessarily constant during the trip.

The work-kinetic energy theorem states that $\Delta K = K_f - K_i = W_{\text{net}}$.

We also have that $K_i = K_f \rightarrow \Delta K = 0 \rightarrow W_{\text{net}} = 0$. There are two forces acting on the object: The gravitational force $F_g$ and the applied force $F$ that lifts the object. $W_{\text{net}} = W_a (A \rightarrow B) + W_g (A \rightarrow B) \rightarrow 0 \rightarrow$

$W_a (A \rightarrow B) - W_g (A \rightarrow B)$

$W_g (A \rightarrow B) mgd \cos 180^\circ = -mgd \rightarrow W_a (A \rightarrow B) mgd$.

Work Done by a Force in Lowering an Object:
In this case the object moves from $B$ to $A$.

$W_g (B \rightarrow A) mgd \cos 0^\circ = mgd \quad W_a (B \rightarrow A) - W_g (B \rightarrow A) = -mgd$
Work Done by a Variable Force $F(x)$ Acting Along the $x$-Axis:

A force $F$ that is not constant but instead varies as a function of $x$ is shown in fig. a. We wish to calculate the work $W$ that $F$ does on an object it moves from position $x_i$ to position $x_f$.

We partition the interval $(x_i, x_f)$ into $N$ "elements" of length $\Delta x$ each, as is shown in fig. b. The work done by $F$ in the $j$th interval is $\Delta W_j = F_{j, \text{avg}} \Delta x$, where $F_{j, \text{avg}}$ is the average value of $F$ over the $j$-th element. $W = \sum_{j=1}^{N} F_{j, \text{avg}} \Delta x$. We then take the limit of the sum as $\Delta x \to 0$, (or equivalently $N \to \infty$).

$$W = \lim_{\Delta x \to 0} \sum_{j=1}^{N} F_{j, \text{avg}} \Delta x = \int_{x_i}^{x_f} F(x) dx.$$  Geometrically, $W$ is the area between the $F(x)$ curve and the $x$-axis, between $x_i$ and $x_f$ (shaded blue in fig. d).

$$W = \int_{x_i}^{x_f} F(x) dx \quad (7-8)$$
The Spring Force:
Fig. a shows a spring in its relaxed state. In fig. b we pull one end of the spring and stretch it by an amount $d$. The spring resists by exerting a force $F$ on our hand in the opposite direction.

In fig. c we push one end of the spring and compress it by an amount $d$. Again the spring resists by exerting a force $F$ on our hand in the opposite direction.

The force $F$ exerted by the spring on whatever agent (in the picture it is our hand) is trying to change its natural length either by extending or by compressing it is given by the equation $F = -kx$. Here $x$ is the amount by which the spring has been extended or compressed. This equation is known as "Hooke's law" and $k$ is known as the "spring constant" $F = -kx$. 

(7-9)
Work Done by a Spring Force:
Consider the relaxed spring of spring constant $k$ shown in (a). By applying an external force we change the spring's length from $x_i$ (see b) to $x_f$ (see c). We will calculate the work $W_s$ done by the spring on the external agent (in this case our hand) that changed the spring length. We assume that the spring is massless and that it obeys Hooke's law.

\[ W_s = \int_{x_i}^{x_f} F(x)dx = \int_{x_i}^{x_f} -kx dx = -k \int_{x_i}^{x_f} x dx. \]

\[ W_s = -k \left[ \frac{x^2}{2} \right]_{x_i}^{x_f} = \frac{kx_i^2}{2} - \frac{kx_f^2}{2}. \]

Quite often we start with a relaxed spring ($x_i = 0$) and we either stretch or compress the spring by an amount $x$ ($x_f = \pm x$). In this case \( W_s = -\frac{kx^2}{2}. \)  
(7-10)
Three-Dimensional Analysis:

In the general case the force $\vec{F}$ acts in three-dimensional space and moves an object on a three-dimensional path from an initial point $A$ to a final point $B$.

The force has the form

$$\vec{F} = F_x \hat{i} + F_y \hat{j} + F_z \hat{k}.$$

Points $A$ and $B$ have coordinates $(x_i, y_i, z_i)$ and $(x_f, y_f, z_f)$ respectively.

$$dW = \vec{F} \cdot d\vec{r} = F_x dx + F_y dy + F_z dz$$

$$W = \int_A^B dW = \int_A^{x_f} F_x dx + \int_{y_i}^{y_f} F_y dy + \int_{z_i}^{z_f} F_z dz$$

(7-11)
Work - Kinetic Energy Theorem with a Variable Force:
Consider a variable force $F(x)$ that moves an object of mass $m$ from point $A$ ($x = x_i$) to point $B$ ($x = x_f$). We apply Newton's second law: $F = ma = m\frac{dv}{dt}$. We then multiply both sides of the last equation with $dx$ and get $Fdx = m\frac{dv}{dt}dx$.

We integrate both sides over $dx$ from $x_i$ to $x_f$:

$$
\int_{x_i}^{x_f} Fdx = \int_{x_i}^{x_f} m\frac{dv}{dt}dx.
$$

Thus the integral becomes:

$$
\left[ \frac{dv}{dt} ight]_{x_i}^{x_f} = \frac{mv_f^2}{2} - \frac{mv_i^2}{2} = K_f - K_i = \Delta K.
$$

Note: The work-kinetic energy theorem has exactly the same form as in the case when $F$ is constant!
Power

We define "power" $P$ as the rate at which work is done by a force $F$. If $F$ does work $W$ in a time interval $\Delta t$ then we define the average power as

$$P_{\text{avg}} = \frac{W}{\Delta t}$$

The instantaneous power is defined as

$$P = \frac{dW}{dt}$$

Unit of $P$: The SI unit of power is the watt. It is defined as the power of an engine that does work $W = 1$ J in a time $t = 1$ second.

A commonly used non-SI power unit is the horsepower (hp), defined as $1$ hp = 746 W.

The kilowatt-hour The kilowatt-hour (kWh) is a unit of work. It is defined as the work performed by an engine of power $P = 1000$ W in a time $t = 1$ hour,

$$W = Pt = 1000 \times 3600 = 3.60 \times 10^6 \text{ J.}$$

The kWh is used by electrical utility companies (check your latest electric bill).
Consider a force $F$ acting on a particle at an angle $\phi$ to the motion. The rate at which $F$ does work is given by $P = \frac{dW}{dt} = \frac{F \cos \phi \, dx}{dt} = F \cos \phi \frac{dx}{dt} = F v \cos \phi$.

$$P = Fv \cos \phi = \vec{F} \cdot \vec{v}$$