Chapter 9

Center of Mass and Linear Momentum

In this chapter we will introduce the following new concepts:

- Center of mass (com) for a system of particles
- The velocity and acceleration of the center of mass
- Linear momentum for a single particle and a system of particles

We will derive the equation of motion for the center of mass, and discuss the principle of conservation of linear momentum.

Finally, we will use the conservation of linear momentum to study collisions in one and two dimensions and derive the equation of motion for rockets.
The Center of Mass:

Consider a system of two particles of masses $m_1$ and $m_2$ at positions $x_1$ and $x_2$, respectively. We define the position of the center of mass (com) as follows:

$$x_{\text{com}} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

We can generalize the above definition for a system of $n$ particles as follows:

$$x_{\text{com}} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3 + \ldots + m_n x_n}{m_1 + m_2 + m_3 + \ldots + m_n} = \frac{1}{M} \sum_{i=1}^{n} m_i x_i$$

Here $M$ is the total mass of all the particles $M = m_1 + m_2 + m_3 + \ldots + m_n$.

We can further generalize the definition for the center of mass of a system of particles in three-dimensional space. We assume that the $i$th particle (mass $m_i$) has position vector $\vec{r}_i$.

$$\vec{r}_{\text{com}} = \frac{1}{M} \sum_{i=1}^{n} m_i \vec{r}_i$$

(9-2)
The position vector for the center of mass is given by the equation \( \vec{r}_{\text{com}} = \frac{1}{M} \sum_{i=1}^{n} m_i \vec{r}_i \).

The position vector can be written as \( \vec{r}_{\text{com}} = x_{\text{com}} \hat{i} + y_{\text{com}} \hat{j} + z_{\text{com}} \hat{k} \).

The components of \( \vec{r}_{\text{com}} \) are given by the equations

\[
x_{\text{com}} = \frac{1}{M} \sum_{i=1}^{n} m_i x_i \quad y_{\text{com}} = \frac{1}{M} \sum_{i=1}^{n} m_i y_i \quad z_{\text{com}} = \frac{1}{M} \sum_{i=1}^{n} m_i z_i
\]

The center of mass has been defined using the equations given above so that it has the following property:

The center of mass of a system of particles moves as though all the system's mass were concentrated there, and that the vector sum of all the external forces were applied there.

The above statement will be proved later. An example is given in the figure. A baseball bat is flipped into the air and moves under the influence of the gravitation force. The center of mass is indicated by the black dot. It follows a parabolic path as discussed in Chapter 4 (projectile motion). All the other points of the bat follow more complicated paths.
The Center of Mass for Solid Bodies:

Solid bodies can be considered as systems with continuous distribution of matter. The sums that are used for the calculation of the center of mass of systems with discrete distribution of mass become integrals:

\[
x_{com} = \frac{1}{M} \int x \, dm \\
y_{com} = \frac{1}{M} \int y \, dm \\
z_{com} = \frac{1}{M} \int z \, dm
\]

The integrals above are rather complicated. A simpler special case is that of uniform objects in which the mass density \( \rho = \frac{dm}{dV} \) is constant and equal to \( \frac{M}{V} \):

\[
x_{com} = \frac{1}{V} \int x \, dV \\
y_{com} = \frac{1}{V} \int y \, dV \\
z_{com} = \frac{1}{V} \int z \, dV
\]

In objects with symmetry elements (symmetry point, symmetry line, symmetry plane) it is not necessary to evaluate the integrals. The center of mass lies on the symmetry element. For example, the com of a uniform sphere coincides with the sphere center. In a uniform rectangular object the com lies at the intersection of the diagonals.
Consider a system of \( n \) particles of masses \( m_1, m_2, m_3, \ldots, m_n \) and position vectors \( \vec{r}_1, \vec{r}_2, \vec{r}_3, \ldots, \vec{r}_n \), respectively. The position vector of the center of mass is given by

\[
M \vec{r}_{\text{com}} = m_1 \vec{r}_1 + m_2 \vec{r}_2 + m_3 \vec{r}_3 + \ldots + m_n \vec{r}_n.
\]

We take the time derivative of both sides

\[
M \frac{d}{dt} \vec{r}_{\text{com}} = m_1 \frac{d}{dt} \vec{r}_1 + m_2 \frac{d}{dt} \vec{r}_2 + m_3 \frac{d}{dt} \vec{r}_3 + \ldots + m_n \frac{d}{dt} \vec{r}_n
\]

\[
M \vec{v}_{\text{com}} = m_1 \vec{v}_1 + m_2 \vec{v}_2 + m_3 \vec{v}_3 + \ldots + m_n \vec{v}_n.
\]

Here \( \vec{v}_{\text{com}} \) is the velocity of the com and \( \vec{v}_i \) is the velocity of the \( i \)th particle. We take the time derivative once more

\[
M \frac{d}{dt} \vec{v}_{\text{com}} = m_1 \frac{d}{dt} \vec{v}_1 + m_2 \frac{d}{dt} \vec{v}_2 + m_3 \frac{d}{dt} \vec{v}_3 + \ldots + m_n \frac{d}{dt} \vec{v}_n
\]

\[
M \vec{a}_{\text{com}} = m_1 \vec{a}_1 + m_2 \vec{a}_2 + m_3 \vec{a}_3 + \ldots + m_n \vec{a}_n.
\]

Here \( \vec{a}_{\text{com}} \) is the acceleration of the com and \( \vec{a}_i \) is the acceleration of the \( i \)th particle.

\[(9-5)\]
We apply Newton's second law for the $i$th particle:
\[ m_i \ddot{a}_i = \vec{F}_i. \] Here $\vec{F}_i$ is the net force on the $i$th particle,
\[ M\ddot{\vec{a}}_{\text{com}} = \sum F_i. \]

The force $\vec{F}_i$ can be decomposed into two components: applied and internal:
\[ \vec{F}_i = \vec{F}_i^{\text{app}} + \vec{F}_i^{\text{int}}. \]

The above equation takes the form:
\[ M\ddot{\vec{a}}_{\text{com}} = \vec{F}^{\text{app}}_{1} + \vec{F}^{\text{int}}_{1} + \vec{F}^{\text{app}}_{2} + \vec{F}^{\text{int}}_{2} + \vec{F}^{\text{app}}_{3} + \vec{F}^{\text{int}}_{3} + \ldots + \vec{F}^{\text{app}}_{n} + \vec{F}^{\text{int}}_{n} \rightarrow \]
\[ M\ddot{\vec{a}}_{\text{com}} = (\vec{F}^{\text{app}}_{1} + \vec{F}^{\text{app}}_{2} + \vec{F}^{\text{app}}_{3} + \ldots + \vec{F}^{\text{app}}_{n}) + (\vec{F}^{\text{int}}_{1} + \vec{F}^{\text{int}}_{2} + \vec{F}^{\text{int}}_{3} + \ldots + \vec{F}^{\text{int}}_{n}) \]

The sum in the first set of parentheses on the RHS of the equation above is just $\vec{F}_{\text{net}}$.

The sum in the second set of parentheses on the RHS vanishes by virtue of Newton's third law.

The equation of motion for the center of mass becomes \[ M\ddot{\vec{a}}_{\text{com}} = \vec{F}_{\text{net}}. \]

In terms of components we have:
\[ F_{\text{net},x} = Ma_{\text{com},x} \quad F_{\text{net},y} = Ma_{\text{com},y} \quad F_{\text{net},z} = Ma_{\text{com},z} \] (9-6)
The equations above show that the center of mass of a system of particles moves as though all the system's mass were concentrated there, and that the vector sum of all the external forces were applied there. A dramatic example is given in the figure. In a fireworks display a rocket is launched and moves under the influence of gravity on a parabolic path (projectile motion). At a certain point the rocket explodes into fragments. If the explosion had not occurred, the rocket would have continued to move on the parabolic trajectory (dashed line). The forces of the explosion, even though large, are all internal and as such cancel out. The only external force is that of gravity and this remains the same before and after the explosion. This means that the center of mass of the fragments follows the same parabolic trajectory that the rocket would have followed had it not exploded.
Linear Momentum:

Linear momentum $\vec{p}$ of a particle of mass $m$ and velocity $\vec{v}$ is defined as $\vec{p} = m\vec{v}$.

The SI unit for linear momentum is the kg.m/s.

Below we will prove the following statement: The time rate of change of the linear momentum of a particle is equal to the magnitude of net force acting on the particle and has the direction of the force.

In equation form: $\vec{F}_{\text{net}} = \frac{d\vec{p}}{dt}$. We will prove this equation using Newton's second law:

$$\vec{p} = m\vec{v} \implies \frac{d\vec{p}}{dt} = \frac{d}{dt} (m\vec{v}) = m\frac{d\vec{v}}{dt} = m\vec{a} = \vec{F}_{\text{net}}$$

This equation is stating that the linear momentum of a particle can be changed only by an external force. If the net external force is zero, the linear momentum cannot change:

$$\vec{F}_{\text{net}} = \frac{d\vec{p}}{dt} \quad (9-8)$$
The Linear Momentum of a System of Particles

In this section we will extend the definition of linear momentum to a system of particles. The $i$th particle has mass $m_i$, velocity $\vec{v}_i$, and linear momentum $\vec{p}_i$.

We define the linear momentum of a system of $n$ particles as follows:

$$\vec{P} = \vec{p}_1 + \vec{p}_2 + \vec{p}_3 + ... + \vec{p}_n = m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3 + ... + m_n\vec{v}_n = M\vec{v}_{\text{com}}.$$  

The linear momentum of a system of particles is equal to the product of the total mass $M$ of the system and the velocity $\vec{v}_{\text{com}}$ of the center of mass.

The time rate of change of $\vec{P}$ is

$$\frac{d\vec{P}}{dt} = \frac{d}{dt}(M\vec{v}_{\text{com}}) = M\vec{a}_{\text{com}} = \vec{F}_{\text{net}}.$$  

The linear momentum $\vec{P}$ of a system of particles can be changed only by a net external force $\vec{F}_{\text{net}}$. If the net external force $\vec{F}_{\text{net}}$ is zero, $\vec{P}$ cannot change.

$$\vec{P} = \vec{p}_1 + \vec{p}_2 + \vec{p}_3 + ... + \vec{p}_n = M\vec{v}_{\text{com}}$$

$$\frac{d\vec{P}}{dt} = \vec{F}_{\text{net}}$$  \hspace{1cm} (9-9)
Collision and Impulse:
We have seen in the previous discussion that the momentum of an object can change if there is a nonzero external force acting on the object. Such forces exist during the collision of two objects. These forces act for a brief time interval, they are large, and they are responsible for the changes in the linear momentum of the colliding objects.

Consider the collision of a baseball with a baseball bat. The collision starts at time $t_i$ when the ball touches the bat and ends at $t_f$ when the two objects separate.

The ball is acted upon by a force $\vec{F}(t)$ during the collision. The magnitude $F(t)$ of the force is plotted versus $t$ in fig. a. The force is nonzero only for the time interval $t_i < t < t_f$.

$$\vec{F}(t) = \frac{d\vec{p}}{dt}.$$ Here $\vec{p}$ is the linear momentum of the ball,

$$d\vec{p} = \vec{F}(t)dt \rightarrow \int_{t_i}^{t_f} d\vec{p} = \int_{t_i}^{t_f} \vec{F}(t)dt \quad (9-10)$$
\[ \int_{t_i}^{t_f} \vec{F}(t) dt = \int_{t_i}^{t_f} \vec{p} = \Delta \vec{p} = \text{change in momentum} \]

\[ \int_{t_i}^{t_f} \vec{F}(t) dt \text{ is known as the impulse } \vec{J} \text{ of the collision.} \]

\[ \vec{J} = \int_{t_i}^{t_f} \vec{F}(t) dt \]

The magnitude of \( \vec{J} \) is equal to the area under the \( F \) versus \( t \) plot of fig. a \( \rightarrow \Delta \vec{p} = \vec{J} \).

In many situations we do not know how the force changes with time but we know the average magnitude \( F_{\text{ave}} \) of the collision force. The magnitude of the impulse is given by

\[ J = F_{\text{ave}} \Delta t, \text{ where } \Delta t = t_f - t_i. \]

Geometrically this means that the area under the \( F \) versus \( t \) plot (fig. a) is equal to the area under the \( F_{\text{ave}} \) versus \( t \) plot (fig. b).

(9-11)
Series of Collisions

Consider a target that collides with a steady stream of identical particles of mass $m$ and velocity $\vec{v}$ along the $x$-axis. A number $n$ of the particles collides with the target during a time interval $\Delta t$. Each particle undergoes a change $\Delta p$ in momentum due to the collision with the target. During each collision a momentum change $-\Delta p$ is imparted on the target. The impulse on the target during the time interval $\Delta t$ is $J = -n\Delta p$.

The average force on the target is $F_{\text{ave}} = \frac{J}{\Delta t} = \frac{-n\Delta p}{\Delta t} = -\frac{n}{\Delta t} m\Delta v$.

Here $\Delta v$ is the change in the velocity of each particle along the $x$-axis due to the collision with the target $\rightarrow F_{\text{ave}} = -\frac{\Delta m}{\Delta t} \Delta v$.

Here $\frac{\Delta m}{\Delta t}$ is the rate at which mass collides with the target.

If the particles stop after the collision, then $\Delta v = 0 - v = -v$.

If the particles bounce backwards, then $\Delta v = -v - v = -2v$.  

(9-12)
Conservation of Linear Momentum:

Consider a system of particles for which \( \vec{F}_{\text{net}} = 0 \)

\[
\frac{d\vec{P}}{dt} = \vec{F}_{\text{net}} = 0 \rightarrow \vec{P} = \text{Constant}
\]

If no net external force acts on a system of particles, the total linear momentum \( \vec{P} \) cannot change.

\[
\begin{bmatrix}
\text{total linear momentum} \\
\text{at some initial time } t_i
\end{bmatrix} =
\begin{bmatrix}
\text{total linear momentum} \\
\text{at some later time } t_f
\end{bmatrix}
\]

The conservation of linear momentum is an important principle in physics. It also provides a powerful rule we can use to solve problems in mechanics such as collisions.

Note 1: In systems in which \( \vec{F}_{\text{net}} = 0 \) we can always apply conservation of linear momentum even when the internal forces are very large as in the case of colliding objects.

Note 2: We will encounter problems (e.g., inelastic collisions) in which the energy is not conserved but the linear momentum is.  

(9-13)
Momentum and Kinetic Energy in Collisions

Consider two colliding objects with masses $m_1$ and $m_2$, initial velocities $\vec{v}_{1i}$ and $\vec{v}_{2i}$, and final velocities $\vec{v}_{1f}$ and $\vec{v}_{2f}$, respectively.

If the system is isolated, i.e., the net force $\vec{F}_{\text{net}} = 0$, linear momentum is conserved. The conservation of linear momentum is true regardless of the collision type. This is a powerful rule that allows us to determine the results of a collision without knowing the details. Collisions are divided into two broad classes: elastic and inelastic.

A collision is elastic if there is no loss of kinetic energy, i.e., $K_i = K_f$.

A collision is inelastic if kinetic energy is lost during the collision due to conversion into other forms of energy. In this case we have $K_f < K_i$.

A special case of inelastic collisions are known as completely inelastic. In these collisions the two colliding objects stick together and they move as a single body. In these collisions the loss of kinetic energy is maximum.
In these collisions the linear momentum of the colliding objects is conserved \( \vec{p}_{1i} + \vec{p}_{2i} = \vec{p}_{1f} + \vec{p}_{2f} \).

\[ m_1 \vec{v}_{1i} + m_1 \vec{v}_{2i} = m_1 \vec{v}_{1f} + m_1 \vec{v}_{2f} \]

**One-Dimensional Inelastic Collisions:**

In these collisions the two colliding objects stick together and move as a single body. In the figure to the left we show a special case in which \( \vec{v}_{2i} = 0 \). \( m_1 v_{1i} = m_1 V + m_1 V \rightarrow V = \frac{m_1}{m_1 + m_2} v_{1i} \)

The velocity of the center of mass in this collision is \( \vec{v}_{\text{com}} = \frac{\vec{P}}{m_1 + m_2} = \frac{\vec{p}_{1i} + \vec{p}_{2i}}{m_1 + m_2} = \frac{m_1 \vec{v}_{1i}}{m_1 + m_2} \).

In the picture to the left we show some freeze-frames of a totally inelastic collision.
One - Dimensional Elastic Collisions

Consider two colliding objects with masses $m_1$ and $m_2$, initial velocities $\vec{v}_{1i}$ and $\vec{v}_{2i}$, and final velocities $\vec{v}_{1f}$ and $\vec{v}_{2f}$, respectively.

Both linear momentum and kinetic energy are conserved.

**Linear momentum conservation:**

$$m_1 v_{1i} + m_1 v_{2i} = m_1 v_{1f} + m_1 v_{2f} \quad \text{(eq. 1)}$$

**Kinetic energy conservation:**

$$\frac{m_1 v_{1i}^2}{2} + \frac{m_1 v_{2i}^2}{2} = \frac{m_1 v_{1f}^2}{2} + \frac{m_2 v_{2f}^2}{2} \quad \text{(eq. 2)}$$

We have two equations and two unknowns, $v_{1f}$ and $v_{1i}$.

If we solve equations 1 and 2 for $v_{1f}$ and $v_{1i}$, we get the following solutions:

$$v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i} + \frac{2m_2}{m_1 + m_2} v_{2i}$$

$$v_{2f} = \frac{2m_1}{m_1 + m_2} v_{1i} + \frac{m_2 - m_1}{m_1 + m_2} v_{2i}$$
Special Case of Elastic Collisions - Stationary Target $v_{2i} = 0$:

We substitute $v_{2i} = 0$ in the two solutions for $v_{1f}$ and $v_{1f}$:

$$v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i} + \frac{2m_2}{m_1 + m_2} v_{2i} \rightarrow v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i}$$

$$v_{2f} = \frac{2m_1}{m_1 + m_2} v_{1i} + \frac{m_2 - m_1}{m_1 + m_2} v_{2i} \rightarrow v_{2f} = \frac{2m_1}{m_1 + m_2} v_{1i}$$

Below we examine several special cases for which we know the outcome of the collision from experience.

1. Equal masses $m_1 = m_2 = m$

$$v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i} = \frac{m - m}{m + m} v_{1i} = 0$$

$$v_{2f} = \frac{2m_1}{m_1 + m_2} v_{1i} = \frac{2m}{m + m} v_{1i} = v_{1i}$$

The two colliding objects have exchanged velocities.

(9-17)
2. A massive target \( m_2 \ll m_1 \rightarrow \frac{m_1}{m_2} \quad 1 \)

\[
\begin{align*}
  v_{1f} &= \frac{m_1 - m_2}{m_1 + m_2} v_{1i} = \frac{m_2}{m_1 + m_2} v_{1i} \approx -v_{1i} \\
  v_{2f} &= \frac{2m_1}{m_1 + m_2} v_{1i} = \frac{m_1}{m_1 + m_2} v_{1i} \approx 2 \left( \frac{m_1}{m_2} \right) v_{1i}
\end{align*}
\]

Body 1 (small mass) bounces back along the incoming path with its speed practically unchanged.

Body 2 (large mass) moves forward with a very small speed because \( \frac{m_1}{m_2} \ll 1 \).

(9-18)
2. A massive projectile \[ m_1 \rightarrow m_2 \rightarrow \frac{m_2}{m_1} \]

\[
v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i} = \frac{1 - \frac{m_2}{m_1}}{1 + \frac{m_2}{m_1}} v_{1i} \approx v_{1i}
\]

\[
v_{2f} = \frac{2m_1}{m_1 + m_2} v_{1i} = \frac{2}{1 + \frac{m_2}{m_1}} v_{1i} \approx 2v_{1i}
\]

Body 1 (large mass) keeps on going, scarcely slowed by the collision. Body 2 (small mass) charges ahead at twice the speed of body 1.
Collisions in Two Dimensions:
In this section we will remove the restriction that the colliding objects move along one axis. Instead we assume that the two bodies that participate in the collision move in the $xy$-plane. Their masses are $m_1$ and $m_2$.

The linear momentum of the system is conserved: \[ \vec{p}_{1i} + \vec{p}_{2i} = \vec{p}_{1f} + \vec{p}_{2f}. \]
If the system is elastic the kinetic energy is also conserved: \[ K_{1i} + K_{2i} = K_{1f} + K_{2f}. \]
We assume that $m_2$ is stationary and that after the collision particle 1 and particle 2 move at angles $\theta_1$ and $\theta_2$ with the initial direction of motion of $m_1$. In this case the conservation of momentum and kinetic energy take the form:

- $x$– axis: \[ m_1 v_{1i} = m_1 v_{1f} \cos \theta_1 + m_2 v_{2f} \cos \theta_2 \quad (\text{eq. 1}) \]
- $y$– axis: \[ 0 = -m_1 v_{1f} \sin \theta_1 + m_2 v_{2f} \sin \theta_2 \quad (\text{eq. 2}) \]

\[
\frac{1}{2} m_1 v_{1i}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2 \quad (\text{eq. 3})
\]
We have three equations and seven variables: 
Two masses: $m_1, m_2$; three speeds: $v_{1i}, v_{1f}, v_{2f}$; and two angles: $\theta_1, \theta_2$. If we know the values of four of these parameters we can calculate the remaining three.

(9-20)
Systems with Varying Mass: The Rocket:

A rocket of mass $M$ and speed $v$ ejects mass backwards at a constant rate $\frac{dM}{dt}$. The ejected material is expelled at a constant speed $v_{\text{rel}}$ relative to the rocket. Thus the rocket loses mass and accelerates forward. We will use the conservation of linear momentum to determine the speed $v$ of the rocket.

In figures (a) and (b) we show the rocket at times $t$ and $t + dt$. If we assume that there are no external forces acting on the rocket, linear momentum is conserved

$$p(t) = p(t + dt) \rightarrow Mv = -UdM + M + dM \quad v + dv \quad (\text{eq. 1}).$$

Here $dM$ is a negative number because the rocket's mass decreases with time $t$. $U$ is the velocity of the ejected gases with respect to the inertial reference frame in which we measure the rocket's speed $v$. We use the transformation equation for velocities (Chapter 4) to express $U$ in terms of $v_{\text{rel}}$, which is measured with respect to the rocket: $U = v + dv - v_{\text{rel}}$. We substitute $U$ in equation 1 and we get

$$Mdv = -dMv_{\text{rel}}. \quad (9-21)$$
Using the conservation of linear momentum we derived the equation of motion for the rocket

\[ M \frac{dv}{dt} = -dMv_{\text{rel}} \]  
(eq. 2). We assume that material is ejected from the rocket's nozzle at a constant rate

\[ \frac{dM}{dt} = -R \]  
(eq. 3). Here \( R \) is a constant positive number.

We divide both sides of eq.(2) by \( dt \)  
\[ M \frac{dv}{dt} = - \frac{dM}{dt} v_{\text{rel}} = Rv_{\text{rel}} \rightarrow \]

\[ Ma = Rv_{\text{rel}} \]  
(First rocket equation). Here \( a \) is the rocket's acceleration.

We use equation 2 to determine the rocket's speed as a function of time \( t \) :

\[ dv = -v_{\text{rel}} \frac{dM}{M} \]. We integrate both sides  
\[ \int dv = -v_{\text{rel}} \int_{M_i}^{M_f} \frac{dM}{M} \rightarrow \]

\[ v_f - v_i = -v_{\text{rel}} \ln \frac{M_i}{M_f} = v_{\text{rel}} \ln \frac{M_i}{M_f} = v_{\text{rel}} \ln \frac{M_i}{M_f} \]  
(Second rocket equation)  
\[ (9-22) \]